April 19 Math 2306 sec. 51 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Recall that for $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0),\\
&\vdots \qquad\qquad \vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Unit Impulse

The Dirac delta, denoted by $\delta(t-a)$, models a force of unit magnitude instantaneously applied at time $t=a$.

This is not a function in the usual sense, but it has several useful properties.

- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ for any real number a.
- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)f(t)\,dt = f(a)$ if a is in the domain of the function f (sifting property).
- $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for any constant $a \geq 0$.
- $\displaystyle\frac{d}{dt}\mathscr{U}(t-a) = \delta(t-a)$ in the sense of *distributions*.

Solve the IVP using the Laplace Transform

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest from a position 10 cm above equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

The corresponding IVP for the situation described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

Let $X(s) = \mathscr{L}\{x(t)\}$.

$$\mathscr{L}\{x'' + 6x' + 10x\} = \mathscr{L}\{\delta(t-1)\}$$

$$\mathscr{L}\{x''\} + 6\mathscr{L}\{x'\} + 10\mathscr{L}\{x\} = e^{-1s}$$

$$s^2X(s) - sx(0) - x'(0) + 6\left(sX(s) - x(0)\right) + 10X(s) = e^{-s}$$

$x(0) = 0.1$, $x'(0) = 0$, $x(0) = 0.1$

$$s^2X(s) - 0.1s - 0 + 6\cdot\left(sX(s) - 0.1\right) + 10X(s) = e^{-s}$$

$$(s^2 + 6s + 10)X(s) - 0.1s - 0.6 = e^{-s}$$

$$(s^2 + 6s + 10)X(s) = e^{-s} + 0.1s + 0.6$$

matches $m^2 + 6m + 10$

$$X(s) = \frac{e^{-s}}{s^2 + 6s + 10} + \frac{0.1s + 0.6}{s^2 + 6s + 10}$$

Discriminant $b^2-4ac = 6^2-4(1)(10) = -4 < 0$

Complete the square

$$s^2+6s+9-9+10 = (s+3)^2+1$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + \frac{0.1s+0.6}{(s+3)^2+1}$$

we need $s+3$ in place of as s's

$$0.1s+0.6 = 0.1(s+3-3)+0.6$$

$$= 0.1(s+3)+0.3$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\,\frac{s+3}{(s+3)^2+1} + 0.3\,\frac{1}{(s+3)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-3t}\sin t \qquad = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= e^{-3t}\cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\,e^{-1s}\right\} = f(t-1)\,u(t-1)$$

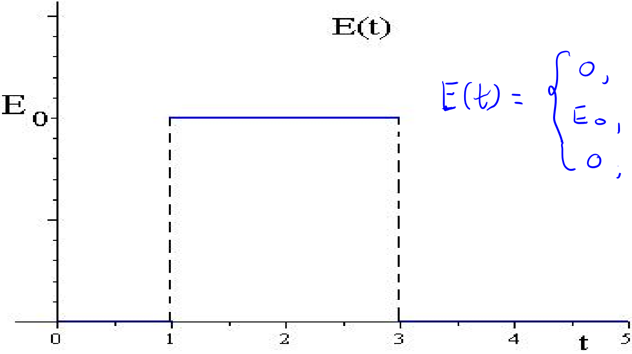
$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\,\frac{s+3}{(s+3)^2+1} + 0.3\,\frac{1}{(s+3)^2+1}$$

$$\boxed{\begin{aligned} &\text{The displacement } x(t) = \mathcal{L}^{-1}\{X(s)\} \\ &x(t) = e^{-3(t-1)}\sin(t-1)\,\mathcal{U}(t-1) + 0.1\,e^{-3t}\cos t \\ &\qquad + 0.3\,e^{-3t}\sin t \end{aligned}}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$Li' + Ri = E$$



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \geq 3 \end{cases}$$

LR Circuit Example

$$1\frac{di}{dt} + 10i = 0 - 0u(t-1) + E_0 u(t-1) - E_0 u(t-3) + 0u(t-3)$$

$$\frac{di}{dt} + 10i = E_0 u(t-1) - E_0 u(t-3)$$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0 u(t-1) - E_0 u(t-3)\}$$

$$\mathcal{L}\{i'\} + 10\mathcal{L}\{i\} = E_0 \mathcal{L}\{u(t-1)\} - E_0 \mathcal{L}\{u(t-3)\}$$

$$sI(s) - \underset{\overset{\shortparallel}{0}}{i(0)} + 10I(s) = E_0 \frac{e^{-s}}{s} - E_0 \frac{e^{-3s}}{s}$$

$$(s+10)I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}$$

Partial fractions

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10} \implies$$

$$1 = A(s+10) + Bs$$

Set $s=0$ $\quad 1 = 10A \quad A = \frac{1}{10}$

$s=-10 \quad 1 = -10B \quad B = \frac{-1}{10}$

$$I(s) = \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right) e^{-s} - \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right) e^{-3s}$$

$$\mathcal{L}^{-1}\left\{ \frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right\} = \frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$f(t) = \frac{E_0}{10} - \frac{E_0}{10} e^{-10t}$$

we'll have, $f(t-1)\mathcal{U}(t-1)$ and $f(t-3)\mathcal{U}(t-3)$

$$I(s) = \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-s} - \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-3s}$$

$$i(t) = \mathcal{L}^{-1}\{I(s)\}$$

$$\boxed{i(t) = \left(\frac{E_0}{10} - \frac{E_0}{10}e^{-10(t-1)}\right)u(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10}e^{-10(t-3)}\right)u(t-3)}$$

This is the current for $t > 0$.

Let's write $i(t)$ in a stacked format

For $t < 1$, $u(t-1) = 0$ and $u(t-3) = 0$

For $1 \le t < 3$, $u(t-1) = 1$ and $u(t-3) = 0$

For $t \geq 3$, $u(t-1) = 1$ and $u(t-3) = 1$

$$\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, \quad - \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-3)} \right)$$

$$\frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}$$

$$\boxed{i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, & 1 \le t < 3 \\ \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}, & t \geq 3 \end{cases}}$$

This is the same i(t) in stacked notation. It is continuous at 1 and 3.