April 19 Math 2306 sec. 52 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Recall that for $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\{y(t)\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0),\\
&\vdots \qquad\qquad \vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Unit Impulse

The Dirac delta, denoted by $\delta(t-a)$, models a force of unit magnitude instantaneously applied at time $t=a$.

This is not a function in the usual sense, but it has several useful properties.

- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ for any real number a.
- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)f(t)\,dt = f(a)$ if a is in the domain of the function f (sifting property).
- $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for any constant $a \geq 0$.
- $\dfrac{d}{dt}\mathscr{U}(t-a) = \delta(t-a)$ in the sense of *distributions*.

Solve the IVP using the Laplace Transform

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest from a position 10 cm above equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

The corresponding IVP for the situation described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$.

$$\mathcal{L}\{x'' + 6x' + 10x\} = \mathcal{L}\{\delta(t-1)\}.$$

$$\mathcal{L}\{x''\} + 6\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = e^{-1s}$$

$$s^2X(s) - sx(0) - x'(0) + 6\left(sX(s) - x(0)\right) + 10X(s) = e^{-s}$$

$$\quad\; 0.1 \qquad\;\; 0 \qquad\qquad\qquad\qquad 0.1$$

$$s^2X(s) - 0.1s + 6sX(s) - 0.6 + 10X(s) = e^{-s}$$

$$(s^2+6s+10)X(s) - 0.1s - 0.6 = e^{-s}$$

$$(s^2+6s+10)X(s) = e^{-s} + 0.1s + 0.6$$

matches $m^2+6m+10$

$$X(s) = \frac{e^{-s}}{s^2+6s+10} + \frac{0.1s+0.6}{s^2+6s+10}$$

$s^2+6s+10$ doesn't factor

$$b^2-4ac = 6^2-4(1)(10) = -4$$

Complete the square

$$s^2+6s+9-9+10 = (s+3)^2+1$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + \frac{0.1s+0.6}{(s+3)^2+1}$$

We need $s+3$ in place of every s

$$0.1s+0.6 = 0.1(s+3-3)+0.6 = 0.1(s+3)+0.3$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\,\frac{s+3}{(s+3)^2+1} + 0.3\,\frac{1}{(s+3)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-3t}\cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-3t}\sin t$$

$$\mathcal{L}^{-1}\left\{e^{-1s}\,\frac{1}{(s+3)^2+1}\right\} = f(t-1)\,u(t-1)$$

where $f(t) = e^{-3t}\sin t$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\,\frac{s+3}{(s+3)^2+1} + 0.3\,\frac{1}{(s+3)^2+1}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

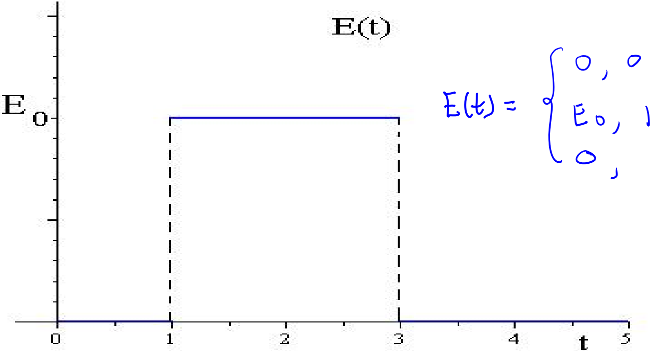
$$x(t) = e^{-3(t-1)}\sin(t-1)\mathcal{U}(t-1) + 0.1e^{-3t}\cos t + 0.3e^{-3t}\sin t$$

The displacement for all $t > 0$

Solve the IVP

An LR-series circuit has inductance $L = 1\text{h}$, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$L\frac{di}{dt} + Ri = E$$



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

LR Circuit Example

$$1\frac{di}{dt}+10i=0-0\mathcal{U}(t-1)+E_0\mathcal{U}(t-1)-E_0\mathcal{U}(t-3)+0\mathcal{U}(t-3)$$

$$\frac{di}{dt}+10i=E_0\mathcal{U}(t-1)-E_0\mathcal{U}(t-3) \qquad i(0)=0$$

Let $I(s)=\mathcal{L}\{i(t)\}$.

$$\mathcal{L}\{i'+10i\}=\mathcal{L}\{E_0\mathcal{U}(t-1)-E_0\mathcal{U}(t-3)\}$$

$$\mathcal{L}\{i'\}+10\mathcal{L}\{i\}=E_0\mathcal{L}\{\mathcal{U}(t-1)\}-E_0\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s)-\underset{\overset{\shortparallel}{0}}{i(0)}+10I(s)=E_0\frac{e^{-s}}{s}-E_0\frac{e^{-3s}}{s}$$

$$(s+10)I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}$$

Decompose $\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$

$$1 = A(s+10) + Bs$$

set $s=0 \qquad 1 = 10A$

$s=-10 \qquad 1 = -10B$

$$\Longrightarrow A = \frac{1}{10}, \quad B = \frac{-1}{10}$$

$$I(s) = \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-s} - \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-3s}$$

Let $f(t) = \mathcal{L}^{-1}\left\{\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right\}$

$$= \frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$f(t) = \frac{E_0}{10} - \frac{E_0}{10}e^{-10t}$$

$$f(t-a)\,u(t-a)$$

$$I(s) = \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-s} - \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right)e^{-3s}$$

The current $i(t) = \mathcal{L}^{-1}\{I(s)\}$

$$\boxed{i(t) = \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}\right) u(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-3)}\right) u(t-3)}$$

when $0 \le t < 1$ $\quad u(t-1) = 0$ and $u(t-3) = 0$

when $1 \le t < 3$, $\quad u(t-1) = 1$ and $u(t-3) = 0$

when $t \geq 3$, $\quad u(t-1) = 1$ and $u(t-3) = 1$

$$\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} - \frac{E_0}{10} + \frac{E_0}{10} e^{-10(t-3)}$$

$$i(t)=\begin{cases} 0, & 0\le t<1 \\ \frac{E_0}{10}-\frac{E_0}{10}e^{-10(t-1)}, & 1\le t<3 \\ \frac{E_0}{10}e^{-10(t-3)}-\frac{E_0}{10}e^{-10(t-1)}, & t\geq 3 \end{cases}$$

This is the same solution i(t) written in the traditional piece-wise defined format.