April 1 Math 3260 sec. 51 Spring 2024

Section 4.5: Dimension of a Vector Space

Theorem:

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

This extends our result in $\mathbb{R}^n$ that said that a set with more vectors than entries in each vector had to be linearly dependent.

For example,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} \right\}$$

is linearly dependent because there are 4 vectors from $\mathbb{R}^3$.

Example

Recall that a basis for $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$.

Is the set below linearly dependent or linearly independent?

$$\{1 + t, 2t - 3t^3, 1 + t + t^2, 1 + t + t^2 + t^3, 2 - t + 2t^3\}$$

A basis has 4 vectors, this set has 5 vectors. Since $5 > 4$, the set is lin. dependent.

All Bases are the same Size

Our theorem gives the immediate corollary:

Corollary:

If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Remark: This makes sense. If one basis had more vectors than another basis, it couldn't be linearly independent.

Dimension

Consider a vector space V.

Definition:

If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[a], then V is said to be **infinite dimensional**.

[a] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Determine $\dim(\mathbb{R}^n)$. $= n$

e.g., the standard basis has n

vectors, $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$.

(b) Determine $\dim \mathrm{Col}(A)$ where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

A is an ref w/ two pivot columns.

A basis would have two elements

so $\dim(\mathrm{Col}(A)) = 2$.

Some Geometry in $\mathbb{R}^3$

We can describe all of the subspaces of $\mathbb{R}^3$ geometrically. The subspace(s) of dimension

(a) zero: is just the origin (one point), $(0, 0, 0)$.

(b) one: are lines through the origin. $\text{Span}\{\mathbf{u}\}$ where $\mathbf{u}$ is not the zero vector.

(c) two: are planes that contain the origin and two other, noncolinear points. $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ with $\{\mathbf{u}, \mathbf{v}\}$ linearly independent.

(d) three: is all of $\mathbb{R}^3$.

Subspaces and Dimension

Theorem:

Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Remark: We said before that we can take a spanning set and remove extra vectors to get a basis. This follow up statement says if we start with a linearly independent set, we can add to it as needed to get a basis.

Subspaces and Dimension

Theorem:

Let V be a vector space with $\dim V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Remark: this connects two properties **spanning** and **linear independence**. If $\dim V = p$ and a set contains p vectors then

- linear independence $\Longrightarrow$ spanning
- spanning $\Longrightarrow$ linear independence

Again, this is **IF** the number of vectors matches the dimension of the vector space.

Column and Null Spaces

Theorem:

Let A be an $m \times n$ matrix. Then

$$\dim \operatorname{Nul} A \ = \ \text{the number of free variables in the equation} \ \ A\mathbf{x} = \mathbf{0},$$

and

$$\dim \operatorname{Col} A \ = \ \text{the number of pivot positions in} \ \ A.$$

Example

A matrix A is show along with its rref. Find the dimensions of the null space and column space of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A has 3 pivot columns so

$\dim(\mathrm{Col}(A)) = 3$

A has one non pivot column hence

$A\vec{x} = \vec{0}$ has one free variable.

$\dim(\mathrm{Nul}(A)) = 1$.

Example

A matrix A along with its rref is shown.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Row A and state $\dim \text{Row}\, A$.

Using the nonzero rows of the rref, a basis for Row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$$

$\dim(\text{Row}(A)) = 3$

Example continued ...

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find a basis for $\operatorname{Col} A$ and state $\dim \operatorname{Col} A$.

The pivot columns are 1, 2, and 4, so a basis for Col (A) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

$\dim(\operatorname{Col}(A)) = 3.$

Example continued ...

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Find a basis for $\mathrm{Nul}\, A$ and state $\dim \mathrm{Nul}\, A$.

From the rref, $A\vec{x} = \vec{0}$

$x_1 = -x_3 - x_5$

$x_2 = 2x_3 - 3x_5$

x_3 - free

$x_4 = 5x_5$

x_5 - free

$$\vec{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

A basis for $\mathrm{Nul}(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

$$\dim(\mathrm{Nul}(A)) = 2$$

Remarks

- Row operations preserve row space, but change linear dependence relations of rows.
- Row operations change column space, but preserve linear dependence relations of columns.
- Another way to obtain a basis for $\mathrm{Row}\, A$ is to take the transpose A^T and do row operations. We have the following relationships:

$$\mathrm{Row}\, A = \mathrm{Col}\, A^T \quad \text{and} \quad \mathrm{Col}\, A = \mathrm{Row}\, A^T$$

Rank & Nullity

Definition:

The **rank** of a matrix A, denoted $\operatorname{rank}(A)$, is the dimension of the column space of A.

Definition:

The **nullity** of a matrix A is the dimension of the null space.

Remark: Since the dimension of the column space is the number of pivot positions, the dimensions of the column and row spaces are the same. That is,

$$\operatorname{rank}(A) = \dim \operatorname{Col}(A) = \dim \operatorname{Row}(A).$$

The Rank-Nullity Theorem

Theorem:

For $m \times n$ matrix A, $\dim \mathrm{Col}(A) = \dim \mathrm{Row}(A) = \mathrm{rank}(A)$. Moreover

$$\mathrm{rank}\, A + \dim \mathrm{Nul}\, A = n.$$

The Rank-Nullity Theorem

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$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Note: This theorem states the rather obvious fact that

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{non-pivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{total number} \\ \text{of columns} \end{array} \right\}.$$

Examples

rank + nullity = n.

(1) A is a 5×4 matrix and $\text{rank}(A) = 4$. What is $\dim \text{Nul}\, A$?

Here, $n = 4$, and the rank is 4.

$$4 + \text{nullity} = 4 \implies \text{nullity} = 0.$$

$$\dim(\text{Nul}(A)) = 0.$$

This means that $\text{Nul}(A) = \{\vec{0}\}$.

Examples

rank + nullity = n

(2) Suppose A is 7×5 and $\dim \operatorname{Col} A = 2$. Determine

1. the nullity of A

   $n=5$, $\text{rank}(A) = \dim(\operatorname{Col}(A)) = 2$

   $2 + \text{nullity} = 5 \Rightarrow \text{nullity} = 3.$

2. the rank of A^T

   $\text{rank}(A^T) = \dim(\operatorname{Col} A^T) = \dim(\operatorname{Row} A)$

   $= \dim(\operatorname{Col} A) = 2$

   $\text{rank}(A^T) = 2$

3. the nullity of A^T

   A^T is 5×7 so A^T has $n = 7$.

   $2 + \text{nullity} = 7 \Rightarrow \text{nullity} = 5.$

Addendum to Invertible Matrix Theorem

Theorem:

Let A be an $n \times n$ matrix. The following are equivalent to the statement that A is invertible.

(m) The columns of A form a basis for $\mathbb{R}^n$

(n) $\operatorname{Col} A = \mathbb{R}^n$

(o) $\dim \operatorname{Col} A = n$

(p) $\operatorname{rank} A = n$

(q) $\operatorname{Nul} A = \{\mathbf{0}\}$

(r) $\dim \operatorname{Nul} A = 0$