April 21 Math 2306 sec. 51 Spring 2023

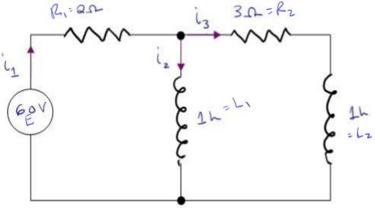
Section 16: Laplace Transforms of Derivatives and IVPs

Solving a System: We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example



$$
\begin{aligned}
L_1 i_2' &+ R_1(i_2+i_3) & &= E \\
L_2 i_3' &+ R_1(i_2+i_3) &+ R_2 i_3 &= E
\end{aligned}
$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$
\begin{aligned}
\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\
\frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0
\end{aligned}
$$

$$\begin{aligned} \frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0 \end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$\mathcal{L}\{x'\} = \mathcal{L}\{-2x - 2y + 60\}$

$\mathcal{L}\{y'\} = \mathcal{L}\{-2x - 5y + 60\}$

$sX(s) - x(0) = -2\mathcal{L}\{x\} - 2\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$

$sY(s) - y(0) = -2\mathcal{L}\{x\} - 5\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$

$$sX - x(0) = -2X - 2Y + \frac{60}{s} \qquad x(0) = 0$$

$$sY - y(0) = -2X - 5Y + \frac{60}{s} \qquad y(0) = 0$$

$$\begin{aligned} sX &= -2X - 2Y + \frac{60}{s} \\ sY &= -2X - 5Y + \frac{60}{s} \end{aligned} \quad \Rightarrow \quad \begin{aligned} sX + 2X + 2Y &= \frac{60}{s} \\ 2X + sY + 5Y &= \frac{60}{s} \end{aligned}$$

$$(s+2)X + 2Y = \frac{60}{s}$$

$$2X + (s+5)Y = \frac{60}{s}$$

$$\begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 60/s \\ 60/s \end{bmatrix}$$

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix}. \qquad \det(A) = (s+2)(s+5) - 4$$

$$= s^2 + 7s + 10 - 4 = s^2 + 7s + 6$$

$$A_x = \begin{bmatrix} 60/s & 2 \\ 60/s & s+5 \end{bmatrix} \qquad \det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s} 2$$

$$= \frac{60}{s}(s+3)$$

$$A_y = \begin{bmatrix} s+2 & 60/s \\ 2 & 60/s \end{bmatrix} \qquad \det(A_y) = \frac{60}{s}(s+2) - \frac{60}{s}(2)$$

$$= \frac{60}{s}(s) = 60$$

$$X(s) = \frac{\det(A_x)}{\det(A)} \qquad Y(s) = \frac{\det(A_y)}{\det(A)}$$

$$X(s) = \frac{\frac{60}{5}(s+3)}{s^2+7s+6} = \frac{60(s+3)}{s(s+1)(s+6)}$$

$$Y(s) = \frac{60}{s^2+7s+6} = \frac{60}{(s+1)(s+6)}$$

Partial fractions

$$X(s) = \frac{60(s+3)}{s(s+1)(s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+6}$$

$A = 30$, $B = -24$, $C = -6$

$$Y(s) = \frac{60}{(s+1)(s+6)} = \frac{D}{s+1} + \frac{E}{s+6}$$

$$D = 12, \quad E = -12$$

$$X(s) = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution to the system

$$x(t) = \mathcal{L}^{-1}\{X(s)\}, \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$x(t) = 30 - 24e^{-t} - 6e^{-6t}$$

$$y(t) = 12e^{-t} - 12e^{-6t}$$