April 21 Math 2306 sec. 52 Spring 2023

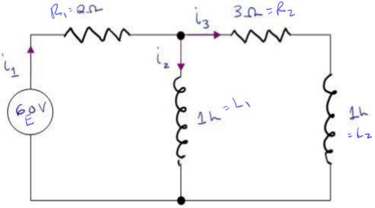
Section 16: Laplace Transforms of Derivatives and IVPs

Solving a System: We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example



$$\begin{aligned} L_1 i_2' &+ R_1(i_2+i_3) & &= E \\ L_2 i_3' &+ R_1(i_2+i_3) &+ R_2 i_3 &= E \end{aligned}$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$\begin{aligned} \frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0 \end{aligned}$$

$$\frac{dx}{dt} = -2x - 2y + 60, \quad x(0) = 0$$
$$\frac{dy}{dt} = -2x - 5y + 60, \quad y(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$, and $Y(s) = \mathcal{L}\{y(t)\}$.

$$\mathcal{L}\{x'\} = \mathcal{L}\{-2x - 2y + 60\} = -2\mathcal{L}\{x\} - 2\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{-2x - 5y + 60\} = -2\mathcal{L}\{x\} - 5\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$sX(s) - \overbrace{x(0)}^{=0} = -2X(s) - 2Y(s) + \frac{60}{s}$$

$$sY(s) - \overbrace{y(0)}^{=0} = -2X(s) - 5Y(s) + 60/s$$

$$sX + 2X + 2Y = \frac{60}{s}$$

$$2X + sY + 5Y = \frac{60}{s}$$

$$(s+2)X + 2Y = \frac{60}{s}$$

$$2X + (s+5)Y = \frac{60}{s}$$

$$\begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 60/s \\ 60/s \end{bmatrix}$$

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \qquad \det(A) = (s+2)(s+5) - 4$$

$$= s^2 + 7s + 10 - 4 = s^2 + 7s + 6$$

$$A_x = \begin{bmatrix} 60/s & 2 \\ 60/s & s+5 \end{bmatrix} \qquad \det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}(2)$$

$$= \frac{60}{s}(s+3)$$

$$A_Y = \begin{bmatrix} s+2 & 60/s \\ 2 & 60/s \end{bmatrix} \qquad \det(A_Y) = (s+2)\frac{60}{s} - (2)\frac{60}{s}$$

$$= \frac{60}{s}(s) = 60$$

$$X(s) = \frac{\det(A_x)}{\det(A)}, \qquad Y(s) = \frac{\det(A_Y)}{\det(A)}$$

$$X(s) = \frac{\frac{60}{s}(s+3)}{s^2+7s+6} = \frac{60(s+3)}{s(s+1)(s+6)}$$

$$Y(s) = \frac{60}{s^2+7s+6} = \frac{60}{(s+1)(s+6)}$$

$$X(s) = \frac{60(s+3)}{s(s+1)\cdot(s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+6}$$

$$A = 30, \quad B = -24, \quad C = -6$$

$$Y(s) = \frac{60}{(s+1)(s+6)} = \frac{D}{s+1} + \frac{E}{s+6}$$

$$D = 12, \quad E = -12$$

$$X(s) = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution $x(t) = \mathcal{L}^{-1}\{X(s)\}$ and $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$x(t) = 30 - 24e^{-t} - 6e^{-6t}$$

$$y(t) = 12e^{-t} - 12e^{-6t}$$

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$$\begin{aligned} \mathcal{L}\{x''\} &= \mathcal{L}\{y\} \\ \mathcal{L}\{y'\} &= \mathcal{L}\{x\} \end{aligned} \Rightarrow \begin{aligned} s^2X - sx(0) - x'(0) &= Y \\ sY - y(0) &= X \end{aligned}$$

$$s^2 X - s = Y \qquad\qquad s^2 X - Y = s$$
$$sY - 1 = X \quad \Longrightarrow \quad -X + sY = 1$$

$$\begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix} \qquad \det(A) = s^3 - 1$$

$$A_X = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \qquad \det(A_X) = s^2 + 1$$

$$A_Y = \begin{bmatrix} s^2 & s \\ -1 & 1 \end{bmatrix} \qquad \det(A_Y) = s^2 + s$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{s^2+1}{s^3-1} = \frac{s^2+1}{(s-1)(s^2+s+1)}$$

$$\varphi(s) = \frac{\det(A_\varphi)}{\det(A)} = \frac{s^2+s}{s^3-1} = \frac{s(s+1)}{(s-1)(s^2+s+1)}$$

We have to do a decomposition and take the inverse transform.
We'll pick this up next time.