April 22 Math 3260 sec. 52 Spring 2022

Section 6.2: Orthogonal Sets

Definition: An indexed set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** provided each pair of distinct vectors in the set is orthogonal. That is, provided

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever} \quad i \neq j.$$

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set.

Orthogonal Basis

Theorem: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then each vector $\mathbf{y}$ in W can be written as the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p, \quad \text{where the weights}$$

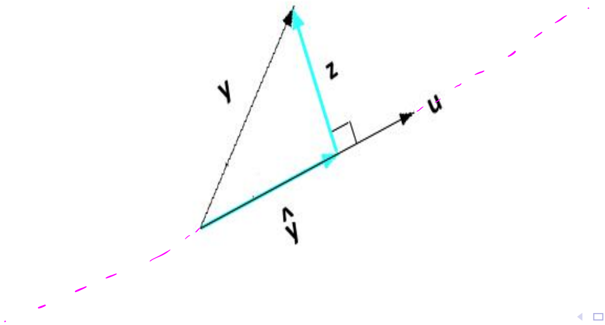
$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}.$$

Projection

Given a nonzero vector $\mathbf{u}$, suppose we wish to decompose another nonzero vector $\mathbf{y}$ into a sum of the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

in such a way that $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$ and $\mathbf{z}$ is perpendicular to $\mathbf{u}$.



Projection

Since $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$, there is a scalar α such that

$$\hat{\mathbf{y}} = \alpha \mathbf{u}.$$

$\vec{y} = \hat{y} + \vec{z}$, $\quad \vec{z} \cdot \vec{u} = 0 \quad$ Find α

$$\vec{u} \cdot \vec{y} = \vec{u} \cdot (\hat{y} + \vec{z}) = \vec{u} \cdot \hat{y} + \underbrace{\vec{u} \cdot \vec{z}}_{= 0}$$

$$\vec{u} \cdot \vec{y} = \vec{u} \cdot (\alpha \vec{u}) = \alpha \, \vec{u} \cdot \vec{u}$$

$$\Rightarrow \quad \alpha = \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}} = \frac{\vec{u} \cdot \vec{y}}{\|\vec{u}\|^2}$$

Projection onto the subspace $L = \text{Span}\{\mathbf{u}\}$

Notation: $\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$

projection of $\vec{y}$ onto L or $\vec{u}$

Example: Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$.

$\vec{y}\cdot\vec{u} = 7(4)+6(2) = 40$, $\vec{u}\cdot\vec{u} = 4^2+2^2 = 20$

$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\vec{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example Continued...

Determine the distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$.

The distance is $\text{dist}(\vec{y}, \hat{y})$

$$\text{dist}(\vec{y}, \hat{y}) = \|\vec{z}\| \qquad \vec{z} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Distance between point and line

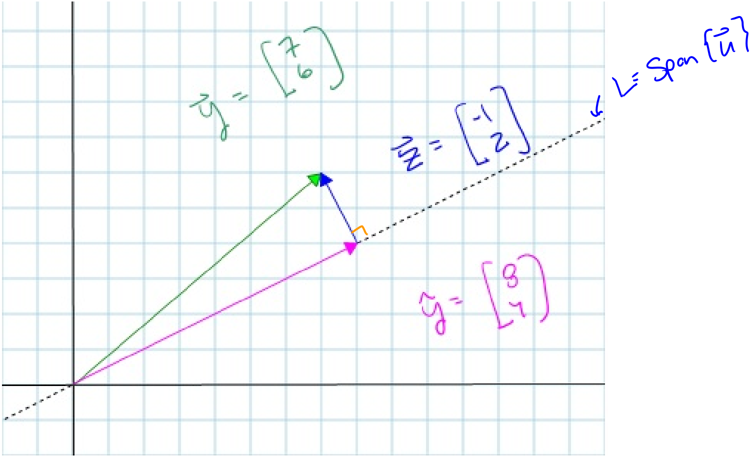


Figure: The distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$ is the norm of $\mathbf{z}$.

Orthonormal Sets

Definition: A set $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is called an **orthonormal set** if it is an orthogonal set of **unit vectors.**

Definition: An **orthonormal basis** of a subspace W of $\mathbb{R}^n$ is a basis that is also an orthonormal set.

Example

The set $\left\{ \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

Note that if $\mathbf{u}_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$, then

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_1 &= \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1 \\ \mathbf{u}_1 \cdot \mathbf{u}_2 &= \left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_2 &= \left(-\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = 1 \end{aligned}$$

Orthogonal Matrix

Consider the matrix $U = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ whose columns are the vectors in the last example. Compute the product

$$U^T U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U^{-1} = U^T$$

What does this say about U^{-1}?

Orthogonal Matrix

Definition: A square matrix U is called an **orthogonal matrix** if $U^T = U^{-1}$.

Theorem: An $n \times n$ matrix U is orthogonal if and only if it's columns form an orthonormal basis of $\mathbb{R}^n$.

The linear transformation associated to an orthogonal matrix preserves *lenghts* and *angles* in the following sense:

Theorem: Orthogonal Matrices

Let U be an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbb{R}^n$. Then

(a) $\|U\mathbf{x}\| = \|\mathbf{x}\|$

(b) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, in particular

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof of (a)

Show that if U is an $n \times n$ orthogonal matrix and $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

Key properties: ① $U^T = U^{-1}$ ③ $(AB)^T = B^T A^T$

② $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$

We'll show that $\|U\vec{x}\|^2 = \|\vec{x}\|^2$

Note

$$\|U\vec{x}\|^2 = (U\vec{x})^T (U\vec{x})$$
$$= (\vec{x}^T U^T)(U\vec{x})$$

$$= \vec{x}^T \underbrace{U^T U}_{I} \vec{x}$$

$$= \vec{x}^T I \vec{x}$$

$$= \vec{x}^T \vec{x}.$$

$$= \|\vec{x}\|^2$$

$$\Rightarrow \quad \|U\vec{x}\|^2 = \|\vec{x}\|^2$$

$$\Rightarrow \quad \|U\vec{x}\| = \|\vec{x}\|$$