April 24 Math 2306 sec. 51 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Solving a System: We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at t = 0, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

< ロ > < 同 > < 回 > < 回 >

Example

Use the Laplace transform to solve the system of equations

$$x''(t) = y, \quad x(0) = 1, \quad x'(0) = 0$$

$$y'(t) = x, \quad y(0) = 1$$

Let $\chi(s) = \chi \{\chi(t)\}$ and $\Upsilon(s) = \chi \{\chi(t)\}$.

$$\chi \{\chi'' \} = \chi \{\chi\} \implies s^{\tau} \chi(s) - s \chi(s) - \chi'(s) = \Upsilon(s)$$

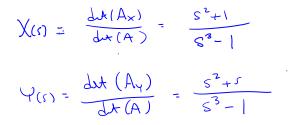
$$\chi \{\chi'\} = \chi \{\chi\} \implies s^{\tau} \chi(s) - g(s) = \chi(s)$$

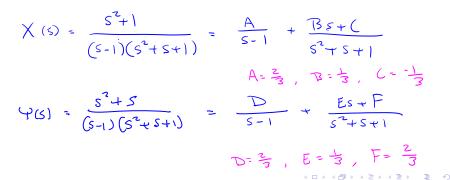
▲ □ ▶ < ⓓ ▶ < 필 ▶ < 필 ▶ < 필 ▶ < 필
 April 21, 2023 2/26

 $S^{2}X - S = \varphi$ $SY - I = X \implies$ 5²X - 4 = 5 -X+5Y=1 In matrix format

 $\begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix}$

 $A = \begin{bmatrix} s^{2} - 1 \\ -1 & s \end{bmatrix} \quad d \neq (A) = s^{3} - 1$ $A_{X} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \quad d \neq (A_{X}) = s^{2} + 1$ $A_{Y} = \begin{bmatrix} s^{2} & s \\ -1 & 1 \end{bmatrix} \quad d \neq (A_{Y}) = s^{2} + s$ $A_{Y} = \begin{bmatrix} s^{2} & s \\ -1 & 1 \end{bmatrix} \quad d \neq (A_{Y}) = s^{2} + s$ $A_{Y} = \begin{bmatrix} s^{2} & s \\ -1 & 1 \end{bmatrix} \quad d \neq (A_{Y}) = s^{2} + s$





April 21, 2023 4/26

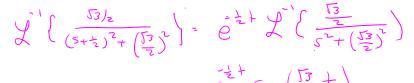
Complete the Square $s^{2} + s + 1 = s^{2} + s + \frac{1}{4} - \frac{1}{4} + 1 = (s + \frac{1}{2})^{2} + \frac{3}{4}$ S++ for need $\chi(s) = \frac{2/3}{s-1} + \frac{1}{3} \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{13}{2})^2} - \frac{1}{13} \frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{13}{2})^2}$ $Y_{00} = \frac{2/3}{5-1} + \frac{1}{3} \frac{5+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{53}{2})^2} + \frac{1}{\sqrt{3}} \frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{1}{2})^2}$

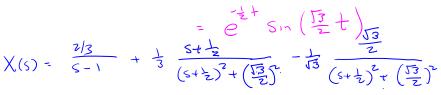
April 21, 2023 5/26

<ロト < 回 > < 回 > < 三 > < 三 > 三 三

 $\mathcal{L}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^{2}+(\frac{1}{2})^{2}}\right\} = \mathcal{C}\left\{\frac{s}{s^{2}+(\frac{1}{2})^{2}}\right\}$

 $= e^{\frac{1}{2}t} \cos\left(\frac{1}{5}t\right)$





 $Y_{(5)} = \frac{2/3}{S-1} + \frac{1}{3} \frac{S+\frac{1}{2}}{(S+\frac{1}{2})^2 + (\frac{53}{2})^2} + \frac{1}{\sqrt{3}} \frac{\frac{53}{2}}{(S+\frac{1}{2})^{2+\frac{3}{2}}(\frac{53}{2})^2}$

X(t)、 言et +方 et Cos(晋七) - 古 et Sin (晋七) $y(t) = = = e^{t} + = e^{t} \cos(\frac{5}{2}t) + = e^{t} \sin(\frac{2}{2}t)$ me solution to the VVP

Convolutions

Consider the problem of evaluating the inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\}.$$
We know that $\frac{1}{s^2+8s+15} = \frac{1}{s+3} \cdot \frac{1}{s+5}$, but since
$$\mathscr{L}^{-1}\left\{F(s)G(s)\right\} \neq \mathscr{L}^{-1}\left\{F(s)\right\} \cdot \mathscr{L}^{-1}\left\{G(s)\right\}$$

writing this product isn't immediately useful. We perform a partial fraction decomposition to write it as a sum.

Remark: There is a meaningful way to evaluate the inverse of a product $\mathscr{L}^{-1}{F(s)G(s)}$. It involves a special kind of product called a **convolution**.

Convolutions

Definition

Let *f* and *g* be piecewise continuous on $[0, \infty)$ and of exponential order. The **convolution** of *f* and *g* is denoted by f * g and is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

イロト イポト イヨト イヨト

April 21, 2023

12/26

Remark: It can readily be shown that f * g = g * f. That is, the convolution is commutative.

Example

Let $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$. Evaluate f * g.

$$(f*g)(t)=\int_0^t f(au)g(t- au)\,d au$$

$$f(\tau) = e^{3\tau}, \quad g(t-\tau) = e^{5(t-\tau)}$$

$$(f * g)(t) = \int_{0}^{t} e^{-3\tau} e^{-5(t-\tau)} d\tau$$

$$= \int_{0}^{t} e^{-3\tau} e^{-5t} e^{5\tau} d\tau$$

$$= e^{-St} \int_{a}^{t} e^{-3t} e^{-5t} dt$$

$$= e^{-St} \int_{a}^{t} e^{-2t} dt$$

$$= e^{-St} \left[\frac{1}{2} e^{2t} \right]_{a}^{t} = e^{-St} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{0} \right]$$

$$= \frac{1}{2} e^{-St} e^{-t} - \frac{1}{2} e^{-St}$$

$$(f * 5) (t) = \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

$$(f (t) = e^{-3t} - \frac{1}{2} e^{-5t}$$

$$(f (t) = e^{-3t} - \frac{1}{2} e^{-5t}$$

Laplace Transforms & Convolutions

Theorem

Suppose
$$\mathscr{L}{f(t)} = F(s)$$
 and $\mathscr{L}{g(t)} = G(s)$. Then

$$\mathscr{L}{f*g} = F(s)G(s)$$

Theorem

Suppose
$$\mathscr{L}^{-1}{F(s)} = f(t)$$
 and $\mathscr{L}^{-1}{G(s)} = g(t)$. Then
 $\mathscr{L}^{-1}{F(s)G(s)} = (f * g)(t)$

イロト イヨト イヨト イヨト April 21, 2023 15/26

Example

Use the convolution to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

$$F(s) = \frac{1}{s+3}, F(s) = \mathcal{L}\left(e^{3t}\right) + f(t) = e^{3t}$$

$$G(s) = \frac{1}{s+5}, G(s) = \mathcal{L}\left(e^{st}\right) + g(t) = e^{st}$$

▲ □ ▶ < ⓓ ▶ < ≧ ▶ < ≧ ▶ ≧
 ✓ ○ ○ ○
 April 21, 2023
 16/26

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$
$$= \left(f * \mathcal{I}\right)(4)$$
$$= \int_{0}^{t} e^{3\tau} e^{-S(t-\tau)} d\tau$$
$$= \int_{0}^{t} e^{-3t} - \frac{1}{2} e^{-St}$$

▲□▶ ▲ 클▶ ▲ 클 ▶ ▲ 클 → 오 ↔
 April 21, 2023 17/26



 $\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)} d\tau\right\}$

$$\int_{0}^{t} \tau^{6} e^{-Y(t-\tau)} d\tau = (f * g)(t)$$

$$f(\tau) = \tau^{6} \implies f(t) = t^{6}$$

$$g(t-\tau) = e^{-Y(t-\tau)} \qquad g(t) = e^{-Yt}$$

<ロト < 昂 > < 臣 > < 臣 > 臣 の Q () April 21, 2023 18/26

$$F(s) = \mathcal{L}\left\{ \left\{ t^{6} \right\} = \frac{6!}{s^{7}} \right\}$$
$$G(s) = \mathcal{L}\left\{ e^{4t} \right\} = \frac{1}{s+4}$$
$$\mathcal{L}\left\{ \int_{0}^{t} \tau^{6} e^{-4(t-\tau)} d\tau \right\} = \frac{6!}{s^{7}} \left(\frac{1}{s+4} \right)$$

 $= \frac{6!}{s^{7}(s+y)}$

- . -

1.000