April 24 Math 2306 sec. 51 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Solving a System: We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$\mathcal{L}\{x''\} = \mathcal{L}\{y\} \Rightarrow s^2 X(s) - s\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=0} = Y(s)$

$\mathcal{L}\{y'\} = \mathcal{L}\{x\} \Rightarrow sY(s) - \underbrace{y(0)}_{=1} = X(s)$

$$s^2X - s = Y \qquad\qquad s^2X - Y = s$$
$$sY - 1 = X \quad\Longrightarrow\quad -X + sY = 1$$

In matrix format

$$\begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix} \qquad \det(A) = s^3 - 1$$

$$A_X = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \qquad \det(A_X) = s^2 + 1$$

$$A_Y = \begin{bmatrix} s^2 & s \\ -1 & 1 \end{bmatrix} \qquad \det(A_Y) = s^2 + s$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{s^2+1}{s^3-1}$$

$$Y(s) = \frac{\det(A_y)}{\det(A)} = \frac{s^2+s}{s^3-1}$$

$$X(s) = \frac{s^2+1}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$$

$A = \frac{2}{3}$, $B = \frac{1}{3}$, $C = \frac{-1}{3}$

$$Y(s) = \frac{s^2+s}{(s-1)(s^2+s+1)} = \frac{D}{s-1} + \frac{Es+F}{s^2+s+1}$$

$D = \frac{2}{3}$, $E = \frac{1}{3}$, $F = \frac{2}{3}$

Complete the square

$$s^2+s+1 = s^2+s+\frac{1}{4}-\frac{1}{4}+1 = \left(s+\frac{1}{2}\right)^2+\frac{3}{4}$$

We also need $s+\frac{1}{2}$ for each s.

$$X(s) = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} = e^{-\frac{1}{2}t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$= e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{3}/2}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} = e^{-\frac{1}{2}t}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{s^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$= e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$X(s) = \frac{2/3}{s-1} + \frac{1}{3}\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1}{3}\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$x(t) = \frac{2}{3} e^{t} + \frac{1}{3} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

$$y(t) = \frac{2}{3} e^{t} + \frac{1}{3} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

The solution to the IVP.

Convolutions

Consider the problem of evaluating the inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\}.$$

We know that $\dfrac{1}{s^2+8s+15}=\dfrac{1}{s+3}\cdot\dfrac{1}{s+5}$, but since

$$\mathscr{L}^{-1}\{F(s)G(s)\}\neq\mathscr{L}^{-1}\{F(s)\}\cdot\mathscr{L}^{-1}\{G(s)\}$$

writing this product isn't immediately useful. We perform a partial fraction decomposition to write it as a sum.

Remark: There is a meaningful way to evaluate the inverse of a product $\mathscr{L}^{-1}\{F(s)G(s)\}$. It involves a special kind of product called a **convolution**.

Convolutions

Definition

Let f and g be piecewise continuous on $[0, \infty)$ and of exponential order. The **convolution** of f and g is denoted by $f * g$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$$

Remark: It can readily be shown that $f * g = g * f$. That is, the convolution is commutative.

Example

Let $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$. Evaluate $f * g$.

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$f(\tau) = e^{-3\tau} \quad , \quad g(t-\tau) = e^{-5(t-\tau)}$$

$$(f*g)(t) = \int_0^t e^{-3\tau} e^{-5(t-\tau)} d\tau$$

$$= \int_0^t e^{-3\tau} e^{-5t} e^{5\tau} d\tau$$

$$= e^{-5t} \int_0^t e^{-3\tau} e^{5\tau} \, d\tau$$

$$= e^{-5t} \int_0^t e^{2\tau} \, d\tau$$

$$= e^{-5t} \left[\frac{1}{2} e^{2\tau} \Big|_0^t = e^{-5t} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{0} \right] \right.$$

$$= \frac{1}{2} e^{-5t} e^{2t} - \frac{1}{2} e^{-5t}$$

$$(f * g)(t) = \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

$$f(t) = e^{-3t}, \quad g(t) = e^{-5t}$$

Laplace Transforms & Convolutions

Theorem

Suppose $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

Theorem

Suppose $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$. Then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Example

Use the convolution to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

$F(s) = \frac{1}{s+3}$, $F(s) = \mathcal{L}\{e^{-3t}\}$ $f(t) = e^{-3t}$

$G(s) = \frac{1}{s+5}$, $G(s) = \mathcal{L}\{e^{-5t}\}$ $g(t) = e^{-5t}$

By our theorem

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

$$= (f * g)(t)$$

$$= \int_0^t e^{-3\tau}\, e^{-5(t-\tau)}\, d\tau$$

$$= \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

Example

$$\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\, d\tau\right\}$$

$$\int_0^t \tau^6 e^{-4(t-\tau)}\, d\tau = (f * g)(t)$$

$$f(\tau) = \tau^6 \Rightarrow f(t) = t^6$$

$$g(t-\tau) = e^{-4(t-\tau)} \quad g(t) = e^{-4t}$$

$$F(s) = \mathcal{L}\{t^6\} = \frac{6!}{s^7}$$

$$G(s) = \mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}$$

$$\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\, d\tau\right\} = \frac{6!}{s^7}\left(\frac{1}{s+4}\right)$$

$$= \frac{6!}{s^7(s+4)}$$