April 24 Math 2306 sec. 52 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Solving a System: We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

We took the transforms letting $X = \mathscr{L}\{x(t)\}$ and $Y = \mathscr{L}\{y(t)\}$, and used Cramer's rule to get to

$$\begin{aligned} X(s) &= \frac{s^2+1}{s^3-1} = \frac{s^2+1}{(s-1)(s^2+s+1)} \\ Y(s) &= \frac{s^2+s}{s^3-1} = \frac{s(s+1)}{(s-1)(s^2+s+1)} \end{aligned}$$

$$X(s) = \frac{s^2+1}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$$

$$\Psi(s) = \frac{s^2+s}{(s-1)(s^2+s+1)} = \frac{D}{s-1} + \frac{Es+F}{s^2+s+1}$$

$A = \frac{2}{3}$, $B = \frac{1}{3}$, $C = \frac{-1}{3}$, $D = \frac{2}{3}$, $E = \frac{1}{3}$, $F = \frac{2}{3}$

Complete the square

$$s^2+s+1 = s^2+s+\tfrac{1}{4}-\tfrac{1}{4}+1 = \left(s+\tfrac{1}{2}\right)^2+\frac{3}{4}$$

$$= \left(s+\tfrac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$X(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\} = e^{-\frac{1}{2}t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\} = e^{-\frac{1}{2}t}\,\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$X(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

The solution $x = \mathcal{L}^{-1}\{X\}$, $y = \mathcal{L}^{-1}\{Y\}$

$$x(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$y(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Convolutions

Consider the problem of evaluating the inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\}.$$

We know that $\dfrac{1}{s^2+8s+15} = \dfrac{1}{s+3}\cdot\dfrac{1}{s+5}$, but since

$$\mathscr{L}^{-1}\{F(s)G(s)\} \neq \mathscr{L}^{-1}\{F(s)\}\cdot\mathscr{L}^{-1}\{G(s)\}$$

writing this product isn't immediately useful. We perform a partial fraction decomposition to write it as a sum.

Remark: There is a meaningful way to evaluate the inverse of a product $\mathscr{L}^{-1}\{F(s)G(s)\}$. It involves a special kind of product called a **convolution**.

Convolutions

Definition

Let f and g be piecewise continuous on $[0, \infty)$ and of exponential order. The **convolution** of f and g is denoted by $f * g$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$$

Remark: It can readily be shown that $f * g = g * f$. That is, the convolution is commutative.

Example

Let $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$. Evaluate $f * g$.

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$f(\tau) = e^{-3\tau}, \quad g(t-\tau) = e^{-5(t-\tau)}$$

$$(f*g)(t) = \int_0^t e^{-3\tau}\, e^{-5(t-\tau)}\,d\tau$$

$$= \int_0^t e^{-3\tau}\, e^{-5t}\, e^{5\tau}\,d\tau$$

$$= e^{-5t} \int_0^t e^{-3\tau} e^{5\tau} \, d\tau$$

$$= e^{-5t} \int_0^t e^{2\tau} \, d\tau$$

$$= e^{-5t} \left[\frac{1}{2} e^{2\tau} \right|_0^t$$

$$= e^{-5t} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{0} \right]$$

$$= \frac{1}{2} e^{-5t} \cdot e^{2t} - \frac{1}{2} e^{-5t}$$

$$(f * g)(t) = \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

for $f(t) = e^{-3t}$, $g(t) = e^{-5t}$

Laplace Transforms & Convolutions

Theorem

Suppose $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

Theorem

Suppose $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$. Then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Example

Use the convolution to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

Let $F(s) = \frac{1}{s+3} = \mathscr{L}\{e^{-3t}\}$

Let $f(t) = e^{-3t}$

$G(s) = \frac{1}{s+5} = \mathscr{L}^{-1}\{e^{-5t}\}$

Let $g(t) = e^{-5t}$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

$$= (f * g)(t)$$

$$= \int_0^t e^{-3\tau} e^{-5(t-\tau)} \, d\tau$$

$$= \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

Example

$$\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\}$$

If $f(\tau) = \tau^6$, $f(t) = t^6$

$g(t-\tau) = e^{-4(t-\tau)} \Rightarrow g(t) = e^{-4t}$

$$\boxed{\mathscr{L}\{f * g\} = F(s)G(s)}$$

Let $F(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{t^6\} = \dfrac{6!}{s^7}$

$G(s) = \mathscr{L}\{g(t)\} = \mathscr{L}\{e^{-4t}\} = \dfrac{1}{s+4}$

$$\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\} = \frac{6!}{s^7}\left(\frac{1}{s+4}\right)$$

$$= \frac{6!}{s^7(s+4)}$$

Example

Use a convolution to evaluate[1]

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+1}\right\}$$

$$= (f*g)(t)$$

where $\mathscr{L}\{f(t)\} = \frac{1}{s^2}$ and $\mathscr{L}\{g(t)\} = \frac{1}{s+1}$

(or vice versa)

$f(t) = t$, $g(t) = e^{-t}$

[1]For comparison, a partial fraction decomp would give
$$\frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}.$$

we need to compute

$$\int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t \tau\, e^{-(t-\tau)}\, d\tau \qquad \text{or}$$

$$\int_0^t g(\tau) f(t-\tau)\, d\tau = \int_0^t e^{-\tau}(t-\tau)\, d\tau$$

Let's do the top one.

$$\int_0^t \tau\, e^{-t} \cdot e^{\tau}\, d\tau = e^{-t} \int_0^t \tau\, e^{\tau}\, d\tau$$

$u = \tau$
$du = d\tau$
$dv = e^{\tau} d\tau$
$v = e^{\tau}$

$$= e^{-t}\left[\tau e^{\tau} \Big|_0^t - \int_0^t e^{\tau}\, d\tau \right]$$

$$= e^{-t}\left[\tau e^{\tau} - e^{\tau}\right|_0^t$$

$$= e^{-t}\left[te^{t} - e^{t} - (0e^{0} - e^{0})\right]$$

$$= e^{-t}\left[te^{t} - e^{t} + 1\right]$$

$$= t - 1 + e^{-t}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = t - 1 + e^{-t}}$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{1}$$

Definition

The function $H(s) = \dfrac{1}{as^2 + bs + c}$ is called the **transfer function** for the differential equation (1).

Definition

The **impulse response** function, $h(t)$, for the differential equation (1) is the inverse Laplace transform of the transfer function

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}.$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t)$$

Remark 1: The transfer function is the Laplace transform of the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Remark 2: The impulse response is the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Convolution

Consider

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Recall the **zero state response** is the inverse transform $\mathscr{L}^{-1}\left\{\dfrac{G(s)}{as^2 + bs + c}\right\}$. Note that we can write this ratio as the product

$$G(s)H(s)$$

where H is the transfer function. If the impulse response is $h(t)$, then the zero state response can be written in terms of a convolution is

$$\mathscr{L}^{-1}\{G(s)H(s)\} = \int_0^t g(\tau)h(t-\tau)\,d\tau$$