April 22 Math 3260 sec. 51 Spring 2024

Section 6.4: Gram-Schmidt Orthogonalization

We saw that the orthogonal decomposition theorem says that if W is a nonzero subspace of \mathbb{R}^n . Each vector **y** in \mathbb{R}^n can be written uniquely as a sum

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and z is in W^{\perp} .

We have a formula for computing $proj_W \mathbf{y}$, but it requires an orthogonal basis for W.

Big Question:

Given any-old basis for a subspace W of \mathbb{R}^n , can we construct an orthogonal basis for that same space?

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Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of \mathbb{R}^n . Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\mathbf{v}_{1} = \mathbf{x}_{1}$$

$$\mathbf{v}_{2} = \mathbf{x}_{2} - \left(\frac{\mathbf{x}_{2} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}}\right) \mathbf{v}_{1}$$

$$\mathbf{v}_{3} = \mathbf{x}_{3} - \left(\frac{\mathbf{x}_{3} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}}\right) \mathbf{v}_{1} - \left(\frac{\mathbf{x}_{3} \cdot \mathbf{v}_{2}}{\mathbf{v}_{2} \cdot \mathbf{v}_{2}}\right) \mathbf{v}_{2}$$

$$\vdots$$

$$P^{-1} \leftarrow P^{-1} \leftarrow P^{-1}$$

$$\mathbf{v}_{\mathcal{P}} = \mathbf{x}_{\mathcal{P}} - \sum_{j=1}^{r} \left(\frac{\mathbf{x}_{\mathcal{P}} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \right) \mathbf{v}_j.$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for *W*. Moreover, for each $k = 1, \ldots, p$

$$\operatorname{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_k\} = \operatorname{Span}\{\mathbf{x}_1,\ldots,\mathbf{x}_k\}.$$

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for $\begin{bmatrix} -1 & 6 & 6 \end{bmatrix}$

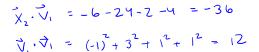
Col A where
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$
. $cre(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

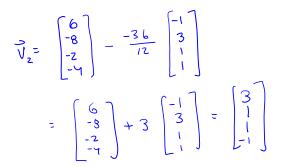
The rolumns form a basis for (al (A).
Let
$$\vec{X}_{1} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$
, $\vec{X}_{2} = \begin{bmatrix} -8 \\ -8 \\ -2 \\ -7 \end{bmatrix}$, $\vec{X}_{3} = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$

Apply Gran Schmidt. $\vec{V}_1 = \vec{X}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

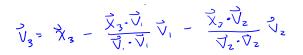
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 $\vec{v}_{z} = \vec{x}_{z} - \vec{x}_{z} \cdot \vec{v}_{i}$





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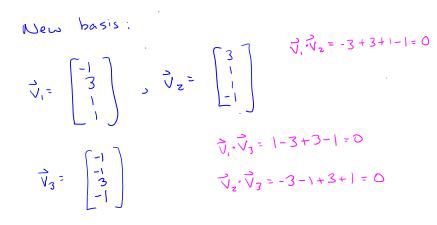


$$\vec{X}_3 \cdot \vec{V}_1 = -6 + 9 + 6 - 3 = 6$$

 $\vec{V}_1 \cdot \vec{V}_1 = 12$
 $\vec{X}_3 \cdot \vec{V}_2 = 18 + 3 + 6 + 3 = 30$
 $\vec{V}_2 \cdot \vec{V}_2 = 3^2 + 1^2 + 1^2 + (-1)^2 = 12$

 $6 + \frac{1}{2} - \frac{15}{2} = 6 - \frac{14}{2} = -1$ 3 - 3 - 5 = 3 - 8 = -1 $6 - \frac{1}{2} - \frac{5}{2} = 6 - \frac{1}{2} = 3$ -3-2+2=-3+2=-1

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An orthogonal basis for (ol(A)
is
$$\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$$

To get an orthonormal basis, we normalize V: 11 V: 11= JIZ for i= 1,2,3 Let $\tilde{W}_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3\\ 1\\ -1 \end{bmatrix}$, $\tilde{W}_2 = \frac{3}{\sqrt{12}} \begin{bmatrix} 3\\ 1\\ -1 \end{bmatrix}$ and $\overline{W}_3 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$. An orthonormal basir is {W, , Wz , Wz].

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