April 22 Math 3260 sec. 51 Spring 2024

Section 6.4: Gram-Schmidt Orthogonalization

We saw that the orthogonal decomposition theorem says that if W is a nonzero subspace of $\mathbb{R}^n$. Each vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as a sum

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

We have a formula for computing $\mathrm{proj}_W\, \mathbf{y}$, but it requires an orthogonal basis for W.

Big Question:

Given any-old basis for a subspace W of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Moreover, for each $k = 1, \ldots, p$

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for

Col A where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The columns form a basis for Col(A).

Let $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$

Apply Gram Schmidt.

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{x}_2 \cdot \vec{v}_1 = -6 - 24 - 2 - 4 = -36$$

$$\vec{v}_1 \cdot \vec{v}_1 = (-1)^2 + 3^2 + 1^2 + 1^2 = 12$$

$$\vec{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$\vec{x}_3 \cdot \vec{v}_1 = -6 + 9 + 6 - 3 = 6$$

$$\vec{v}_1 \cdot \vec{v}_1 = 12$$

$$\vec{x}_3 \cdot \vec{v}_2 = 18 + 3 + 6 + 3 = 30$$

$$\vec{v}_2 \cdot \vec{v}_2 = 3^2 + 1^2 + 1^2 + (-1)^2 = 12$$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$6 + \frac{1}{2} - \frac{15}{2} = 6 - \frac{14}{2} = -1$$

$$3 - \frac{3}{2} - \frac{5}{2} = 3 - \frac{8}{2} = -1$$

$$6 - \frac{1}{2} - \frac{5}{2} = 6 - \frac{6}{2} = 3$$

$$-3 - \frac{1}{2} + \frac{5}{2} = -3 + \frac{4}{2} = -1$$

New basis:

$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = -3 + 3 + 1 - 1 = 0$

$\vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_3 = 1 - 3 + 3 - 1 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = -3 - 1 + 3 + 1 = 0$

An orthogonal basis for Col(A)

is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

To get an orthonormal basis, we normalize $\vec{v}_i$.

$\|\vec{v}_i\| = \sqrt{12}$ for $i = 1, 2, 3$

Let $\vec{w}_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{w}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

and $\vec{w}_3 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$.

An orthonormal basis is $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.