April 25 Math 3260 sec. 52 Spring 2022

Section 6.2: Orthogonal Sets

Definition: (Orthogonal Matrix) A square matrix U is called an **orthogonal matrix** if $U^T = U^{-1}$.

Theorem: An $n \times n$ matrix U is orthogonal if and only if it's columns form an orthonormal basis of $\mathbb{R}^n$.

The linear transformation associated to an orthogonal matrix preserves *lenghts* and *angles* in the following sense:

Theorem: Orthogonal Matrices

Let U be an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ vectors in $\mathbb{R}^n$. Then

(a) $\|U\mathbf{x}\| = \|\mathbf{x}\|$

(b) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, in particular

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Example

Find an orthogonal matrix of the form $U = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & a \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & b \\ \frac{1}{3} & 0 & c \end{bmatrix}$.

Call the columns $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$.

Let's note that $\vec{u}_1$ and $\vec{u}_2$ are orthonormal.

$$\vec{u}_1 \cdot \vec{u}_1 = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{9}{9} = 1$$

$$\vec{u}_2 \cdot \vec{u}_2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{2}{2} = 1$$

$$\vec{u}_1 \cdot \vec{u}_2 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

We need $\vec{u}_1 \cdot \vec{u}_3 = 0$, $\vec{u}_2 \cdot \vec{u}_3 = 0$, $\vec{u}_3 \cdot \vec{u}_3 = 1$

$$\vec{u}_1 \cdot \vec{u}_3 = \frac{2}{3}a + \frac{2}{3}b + \frac{1}{3}c = 0 \implies$$

$$2a + 2b + c = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} + 0 = 0 \implies a = b$$

Combining, $4a + c = 0 \implies c = -4a$

Our column looks like $\begin{bmatrix} a \\ a \\ -4a \end{bmatrix}$.

$$\vec{u}_3 \cdot \vec{u}_3 = a^2 + a^2 + (-4a)^2 = 1 \implies 18a^2 = 1$$

$a^2 = \frac{1}{18}$ one choice is $a = \frac{1}{\sqrt{18}}$

The last column is $\begin{bmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ -4/\sqrt{18} \end{bmatrix}$

$$U = \begin{bmatrix} 2/3 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ 2/3 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ 1/3 & 0 & \frac{-4}{\sqrt{18}} \end{bmatrix}$$

Section 6.3: Orthogonal Projections

Equating points with position vectors, we may wish to find the point $\hat{\mathbf{y}}$ in a subspace W of $\mathbb{R}^n$ that is *closest* to a given point $\mathbf{y}$.

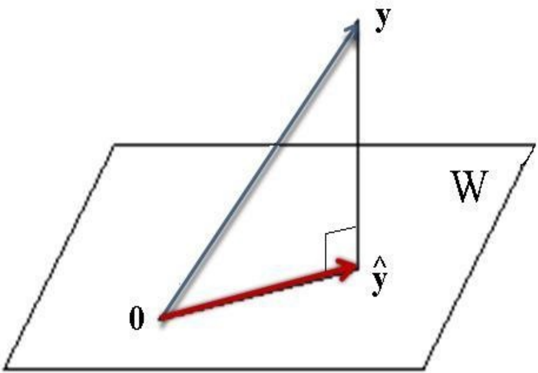

Figure: Illustration of an orthogonal projection. Note that $\text{dist}(\mathbf{y}, \hat{\mathbf{y}})$ is the shortest distance between $\mathbf{y}$ and the points on W.

Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Each vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as a sum

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis for W, then

$$\hat{\mathbf{y}} = \sum_{j=1}^{p} \left(\frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \right) \mathbf{u}_j, \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

The formula for $\hat{\mathbf{y}}$ looks just like the projection onto a line, but with more terms. That is,

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 + \left(\frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \right) \mathbf{u}_2 + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \right) \mathbf{u}_p$$

Orthogonal Decomposition Theorem

Remark 1: Note that the basis must be orthogonal, but otherwise the vector $\hat{\mathbf{y}}$ is **independent** of the particular basis used!

Remark 2: The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of y onto** W. We can denote it

$$\text{proj}_W \, \mathbf{y}.$$

Remark 3: All you really have to do is remember how to project onto a line. Notice that

$$\text{proj}_{\mathbf{u}_1} \mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1.$$

If $W = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ with the **u**'s orthogonal, then

$$\text{proj}_W \, \mathbf{y} = \text{proj}_{\mathbf{u}_1} \mathbf{y} + \text{proj}_{\mathbf{u}_2} \mathbf{y} + \cdots + \text{proj}_{\mathbf{u}_p} \mathbf{y}.$$

Example

Let $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$ and

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}. = \{\vec{w}_1, \vec{w}_2\}$$

(a) Verify that the spanning vectors for W given are an orthogonal basis for W.

$$\vec{w}_1 \cdot \vec{w}_2 = 2(-2) + 1(2) + 2(1) = -4 + 2 + 2 = 0$$

so $\{\vec{w}_1, \vec{w}_2\}$ is orthogonal

Example Continued...

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$$

(b) Find the orthogonal projection of $\mathbf{y}$ onto W.

$$\text{proj}_W \vec{y} = \frac{\vec{y}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\vec{w}_1 + \frac{\vec{y}\cdot\vec{w}_2}{\vec{w}_2\cdot\vec{w}_2}\vec{w}_2$$

$$\vec{y}\cdot\vec{w}_1 = 2(4)+1(8)+2(1) = 18$$

$$\vec{w}_1\cdot\vec{w}_1 = 2^2+1^2+2^2 = 9$$

$$\vec{y}\cdot\vec{w}_2 = -2(4)+2(8)+1(1) = 9$$

$$\vec{w}_2\cdot\vec{w}_2 = (-2)^2+2^2+1 = 9$$

$$\hat{y} = \frac{18}{9}\vec{w}_1 + \frac{9}{9}\vec{w}_2 = 2\vec{w}_1 + 1\vec{w}_2$$

$$= 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

(c) Find the shortest distance between **y** and the subspace W.

$$\text{dist}(\vec{y}, \hat{y}) = \|\vec{z}\| \quad \text{where } \vec{z} = \vec{y} - \hat{y}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}$$

$$\text{dist}(\vec{y}, \hat{y}) = \sqrt{2^2 + 4^2 + (-4)^2} = \sqrt{36} = 6$$

Computing Orthogonal Projections

Theorem: If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis of a subspace W of $\mathbb{R}^n$, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$ then

$$\text{proj}_W\, \mathbf{y} = \sum_{j=1}^{p} \left(\mathbf{y} \cdot \mathbf{u}_j\right) \mathbf{u}_j.$$

And, if U is the matrix $U = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_p]$, then the above is equivalent to

$$\text{proj}_W\, \mathbf{y} = UU^T\mathbf{y}.$$

Remark: In general, U is not square; it's $n \times p$. So even though UU^T will be a square matrix, it is not the same matrix as U^TU and it is not the identity matrix.

Example

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} = \{\vec{w}_1, \vec{w}_2\}$$

Find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for W. Then compute the matrices $U^T U$ and UU^T where $U = [\mathbf{u}_1 \ \ \mathbf{u}_2]$.

From before
$\|\vec{w}_1\| = \|\vec{w}_2\| = 3$

$$\vec{u}_1 = \frac{1}{\|\vec{w}_1\|}\vec{w}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\|\vec{w}_2\|}\vec{w}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$U = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \qquad U^T = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$U^T U = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2\times3 \quad 3\times2$

$$U U^T = \begin{bmatrix} 2/3 & -2/3 \\ 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$3\times2 \quad 2\times3$

$$= \begin{bmatrix} \frac{8}{9} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

$$(uu^T)^T = (u^T)^T u^T = uu^T$$

Example

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$$

Use the matrix formulation to find $\text{proj}_W\, \mathbf{y}$.

$$\text{proj}_W \vec{y} = UU^T \vec{y}$$

$$= \frac{1}{9} \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 18 \\ 36 \\ 45 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$32 - 16 + 2$

$-8 + 40 + 4$

$8 + 32 + 5$