April 26 Math 2306 sec. 51 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

Definition

Let f and g be piecewise continuous on $[0,\infty)$ and of exponential order. The **convolution** of f and g is denoted by $f*g$ and is defined by

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

Remark: It can readily be shown that $f*g=g*f$. That is, the convolution is commutative.

Laplace Transforms & Convolutions

Theorem

Suppose $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

Theorem

Suppose $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$. Then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Example: Use a convolution to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{1}{s+1}\right)\right\}$$

let $F(s) = \frac{1}{s^2}$, $F(s) = \mathscr{L}\{t\}$, $f(t) = t$

$G(s) = \frac{1}{s+1}$, $G(s) = \mathscr{L}\{e^{-t}\}$, $g(t) = e^{-t}$

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

The argument would have partial fraction decomp $\frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$.

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t \tau\, e^{-(t-\tau)}\, d\tau.$$

$$(g*f)(t) = \int_0^t g(\tau)\,f(t-\tau)\,d\tau = \int_0^t e^{-\tau}(t-\tau)\,d\tau$$

we'll do the 1st one.

$$(f*g)(t) = \int_0^t \tau\, e^{-t} e^{\tau}\, d\tau$$

$$= e^{-t}\int_0^t \tau\, e^{\tau}\, d\tau \qquad u=\tau \quad du = d\tau$$

$$v = e^{\tau} \quad dv = e^{\tau} d\tau$$

$$= e^{-t}\left(\tau e^{\tau}\Big|_0^t - \int_0^t e^{\tau}\, d\tau\right)$$

$$= e^{-t}\left(te^{t} - 0e^{0} - e^{\tau}\Big|_0^t\right.$$

$$= e^{-t}\left(te^{t} - e^{t} + e^{0}\right)$$

$$= t - 1 + e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = t - 1 + e^{-t}$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{1}$$

Definition

The function $H(s) = \dfrac{1}{as^2 + bs + c}$ is called the **transfer function** for the differential equation (1).

Definition

The **impulse response** function, $h(t)$, for the differential equation (1) is the inverse Laplace transform of the transfer function

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}.$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t)$$

Remark 1: The transfer function is the Laplace transform of the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Remark 2: The impulse response is the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Convolution

Consider

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Recall the **zero state response** is the inverse transform $\mathscr{L}^{-1}\left\{\dfrac{G(s)}{as^2+bs+c}\right\}$. Note that we can write this ratio as the product

$$G(s)H(s)$$

where H is the transfer function. If the impulse response is $h(t)$, then the zero state response can be written in terms of a convolution is

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\} = \int_0^t g(\tau)h(t-\tau)\,d\tau$$