April 27 Math 3260 sec. 51 Spring 2022

Section 6.4: Gram-Schmidt Orthogonalization

Question: Given any-old basis for a subspace W of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

Example: Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ that spans W.

We need $\vec{v}_1$ and $\vec{v}_2$ in W. So set

$\vec{v}_1 = a_1\vec{x}_1 + a_2\vec{x}_2$ and $\vec{v}_2 = b_1\vec{x}_1 + b_2\vec{x}_2$

We have some freedom on the a_1, a_2, b_1, b_2

Set $a_1 = 1$ and $a_2 = 0$ $\Longrightarrow$ $\vec{v}_1 = \vec{x}_1$

This makes $\text{Span}\{\vec{v}_1\} = \text{Span}\{\vec{x}_1\}$

For $\vec{v}_2$, we need $\vec{x}_2$ so that we set all of W. Let's set $b_2 = 1$.

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = b_1\vec{x}_1 + \vec{x}_2$$

We'll find b_1 by insisting that $\vec{v}_1 \cdot \vec{v}_2 = 0$

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{x}_1 \cdot (b_1\vec{x}_1 + \vec{x}_2) = b_1\vec{x}_1 \cdot \vec{x}_1 + \vec{x}_1 \cdot \vec{x}_2 = 0$$

Solve for b_1

$$b_1\vec{x}_1 \cdot \vec{x}_1 = -\vec{x}_1 \cdot \vec{x}_2 \Rightarrow b_1 = \frac{-\vec{x}_1 \cdot \vec{x}_2}{\vec{x}_1 \cdot \vec{x}_1}.$$

So $\vec{v}_1 = \vec{x}_1$ and $\vec{v}_2 = \vec{x}_2 - \dfrac{\vec{x}_1 \cdot \vec{x}_2}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1$

Since $\vec{x}_1 = \vec{v}_1$ we can write

$$\vec{v}_1 = \vec{x}_1$$
$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$\vec{x}_2 \cdot \vec{v}_1 = 0 - 1 - 1 = -2$, $\vec{x}_1 \cdot \vec{x}_1 = 1 + 1 + 1 = 3$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

New basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \right\}$

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j .
\end{aligned}
$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Moreover, for each $k = 1, \ldots, p$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for

$$\text{Col } A \text{ where } A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

We'll start with a basis for Col (A).

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns are lin. independent. The columns are a basis.

Call them $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$. Let's use Gram-Schmidt to get an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 \cdot \vec{v}_1 = -6 - 24 - 2 - 4 = -36$$

$$\vec{v}_1 \cdot \vec{v}_1 = 1 + 9 + 1 + 1 = 12$$

$$\vec{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$\vec{x}_3 \cdot \vec{v}_1 = -6 + 9 + 6 - 3 = 6 \qquad \vec{v}_1 \cdot \vec{v}_1 = 12$$

$$\vec{x}_3 \cdot \vec{v}_2 = 18 + 3 + 6 + 3 = 30 \qquad \vec{v}_2 \cdot \vec{v}_2 = 12$$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Scratch: $6 + \frac{1}{2} - \frac{15}{2} = 6 - \frac{14}{2} = -1$

$$3 - \frac{3}{2} - \frac{5}{2} = 3 - \frac{8}{2} = -1$$

$$6 - \frac{1}{2} - \frac{5}{2} = 6 - \frac{6}{2} = 3$$

$$-3 - \frac{1}{2} + \frac{5}{2} = -3 + \frac{4}{2} = -1$$

The orthogonal basis is

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

To get orthonormal vectors, we normalize

$$\|\vec{v}_1\|^2 = 12, \quad \|\vec{v}_2\|^2 = 12, \quad \|\vec{v}_3\|^2 = 12$$

$$\Rightarrow \quad \|\vec{v}_i\| = \sqrt{12} = 2\sqrt{3}$$

Let $\vec{w}_i = \dfrac{1}{\|\vec{v}_i\|}\vec{v}_i$

$$\vec{w}_1 = \begin{bmatrix} \frac{-1}{\sqrt{12}} \\ \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} \frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{-1}{\sqrt{12}} \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix}$$

Some Results of Gram-Schmidt Process

- $\mathrm{Span}\{\mathbf{v}_1\}$ is the same space as $\mathrm{Span}\{\mathbf{x}_1\}$, $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the same space as $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, and in general $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is the same space as $\mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$

- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$ where $\mathbf{p}_k$ is the projection of $\mathbf{x}_k$ on the subspace $\mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$

- $\mathbf{v}_k$ is orthogonal to $\mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$, so

- $\|\mathbf{v}_k\|$ is the distance between $\mathbf{x}_k$ and $\mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$

- The process can be used to find an orthonormal basis by either normalizing each vector as it is generated, or by normalizing the orthogonal basis vectors after all have been generated.