April 27 Math 3260 sec. 52 Spring 2022

Section 6.4: Gram-Schmidt Orthogonalization

Question: Given any-old basis for a subspace W of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

Example: Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ that spans W.

We need $\vec{v}_1$ and $\vec{v}_2$ to be in W. Set

$\vec{v}_1 = a_1\vec{x}_1 + a_2\vec{x}_2$ and $\vec{v}_2 = b_1\vec{x}_1 + b_2\vec{x}_2$

We don't really need four variables. Set

$a_1 = 1$ and $a_2 = 0$. $\Longrightarrow$ $\vec{v}_1 = \vec{x}_1$

This gives the bonus $\text{Span}\{\vec{v}_1\} = \text{Span}\{\vec{x}_1\}$.

We have

$$\vec{v}_2 = b_1\vec{x}_1 + b_2\vec{x}_2$$

To make sure we get $\vec{x}_2$, set $b_2 = 1$. We need

$\vec{v}_1 \cdot \vec{v}_2 = 0$

$$\Rightarrow \vec{x}_1 \cdot (b_1\vec{x}_1 + \vec{x}_2) = b_1\vec{x}_1\cdot\vec{x}_1 + \vec{x}_1\cdot\vec{x}_2 = 0$$

Solve for b_1

$$b_1\vec{x}_1\cdot\vec{x}_1 = -\vec{x}_1\cdot\vec{x}_2 \Rightarrow b_1 = \frac{-\vec{x}_1\cdot\vec{x}_2}{\vec{x}_1\cdot\vec{x}_1}$$

We get

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_1\cdot\vec{x}_2}{\vec{x}_1\cdot\vec{x}_1}\vec{x}_1$$

Using $\vec{v}_1 = \vec{x}_1$ we can write

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

For the given $\vec{x}$'s.

$$\vec{x}_2 \cdot \vec{v}_1 = 0 - 1 - 1 = -2, \quad \vec{v}_1 \cdot \vec{v}_1 = 1 + 1 + 1 = 3$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

The new basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \right\}$$

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Moreover, for each $k = 1, \ldots, p$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for

Col A where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

we need to start with a basis for Col(A),

$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Call the columns $\vec{x}_1, \vec{x}_2, \vec{x}_3$

they are all pivot columns.

$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{x}_2 \cdot \vec{v}_1 = -6 - 24 - 2 - 4 = -36, \quad \vec{v}_1 \cdot \vec{v}_1 = 12$$

$$\vec{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

$$\vec{V}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1} \vec{V}_1 - \frac{\vec{x}_3 \cdot \vec{V}_2}{\vec{V}_2 \cdot \vec{V}_2} \vec{V}_2$$

$$\vec{x}_3 \cdot \vec{V}_1 = -6 + 9 + 6 - 3 = 6 \qquad \vec{V}_1 \cdot \vec{V}_1 = 12$$

$$\vec{x}_3 \cdot \vec{V}_2 = 18 + 3 + 6 + 3 = 30 \qquad \vec{V}_2 \cdot \vec{V}_2 = 12$$

$$\vec{V}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Scratch:

$$6 + \frac{1}{2} - \frac{15}{2} = 6 - \frac{14}{2} = -1$$

$$3 - \frac{3}{2} - \frac{5}{2} = 3 - \frac{8}{2} = -1$$

$$6 - \frac{1}{2} - \frac{5}{2} = 6 - \frac{6}{2} = 3$$

$$-3 - \frac{1}{2} + \frac{5}{2} = -3 + \frac{4}{2} = -1$$

The orthogonal basis elements are

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

To get an orthonormal basis, we normalize.

$$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = \sqrt{12}$$

Calling the normalized vectors $\vec{w}_i$

$$\vec{w}_1 = \begin{bmatrix} -1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix}$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthonormal basis for $\mathrm{Col}(A)$.

Some Results of Gram-Schmidt Process

- $\mathrm{Span}\{\mathbf{v}_1\}$ is the same space as $\mathrm{Span}\{\mathbf{x}_1\}$, $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the same space as $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, and in general $\mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is the same space as $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$
- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$ where $\mathbf{p}_k$ is the projection of $\mathbf{x}_k$ on the subspace $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}$
- $\mathbf{v}_k$ is orthogonal to $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}$, so
- $\|\mathbf{v}_k\|$ is the distance between $\mathbf{x}_k$ and $\mathrm{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_{k-1}\}$
- The process can be used to find an orthonormal basis by either normalizing each vector as it is generated, or by normalizing the orthogonal basis vectors after all have been generated.