April 2 Math 2254 sec 002 Spring 2015

Section 11.2: Series

Theorem: The series $a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$ is convergent if $|r| < 1$. In this case,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad |r| < 1.$$

If $|r| \geq 1$, the series is divergent.

A geometric series may appear in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

Strict use of the formula $a/(1-r)$ requires the first term in the sum to be a.

Examples:

Determine the convergence or divergence of the series. If convergent, find the sum.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{7} = \frac{1}{7} + \frac{3}{7} + \frac{9}{7} + \frac{27}{7} + \cdots$

$= \displaystyle\sum_{n=0}^{\infty} \frac{1}{7} 3^n$ $\quad a = \frac{1}{7} \quad r = 3$

$|r| = 3 \geq 1$

The series is divergent.

(b) $\sum_{n=0}^{\infty} \frac{5^{n+1}}{3^{2n-1}}$

$5^{n+1} = 5^n \cdot 5^1 = 5 \cdot 5^n$

$3^{2n-1} = 3^{2n} \cdot 3^{-1} = \frac{1}{3}\left(3^2\right)^n = \frac{1}{3} \cdot 9^n$

$= \sum_{n=0}^{\infty} \frac{5 \cdot 5^n}{\frac{1}{3} \cdot 9^n}$

$a = 15 \qquad r = \frac{5}{9} \qquad |r| = \frac{5}{9} < 1$

Convergent

$= \sum_{n=0}^{\infty} 15 \left(\frac{5}{9}\right)^n$

$= \frac{15}{1 - \frac{5}{9}} = \frac{15}{1 - \frac{5}{9}} \cdot \frac{9}{9} = \frac{135}{9-5} = \frac{135}{4}$

Telescoping Sum

The series $\sum \frac{1}{k(k+1)}$ is an example of a *telescoping* series.

Definition: A series of the form

$$\sum_{k=1}^{\infty} (a_k - a_{k+1})$$

is called a **telescoping series.** The sequence of partial sums is determined to be

$$s_n = a_1 - a_{n+1}$$

and is convergent if and only if $\lim_{n\to\infty} a_n$ exists (as a finite number).

A Special Series: The Harmonic Series

Definition: The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is called the **harmonic series.**

Theorem: The harmonic series is divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = 1 + 1\left(\frac{1}{2}\right)$$

$$\frac{1}{3} > \frac{1}{4}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + 2\left(\frac{1}{2}\right)$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + 3\left(\frac{1}{2}\right)$$

$$S_{16} = 1 + \frac{1}{2} + \dots + \frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{2} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}}_{4} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}}_{8}$$

$$= 1 + 4\left(\frac{1}{2}\right)$$

$$\vdots$$

$$S_{2^n} \geq 1 + n\left(\frac{1}{2}\right)$$

Since $\lim_{n\to\infty}\left(1 + \frac{n}{2}\right) = \infty$, $\quad S_n \to \infty$

Hence the series diverges.

Theorem: (a test for divergence)

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Caution: The converse is NOT true!

Theorem: **(The Divergence Test)**[1] If

$$\lim_{n\to\infty} a_n \quad \text{does not exists, or} \qquad \lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

[1] The Divergence Test is also known as the **n**th **Term Test**.

Example:

Apply the divergence test to the following series. Determine if any conclusion can be made about convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+4}$

$a_n = \dfrac{n}{2n+4}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+4} = \lim_{n\to\infty} \frac{n}{2n+4} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2+\frac{4}{n}} = \frac{1}{2+0} = \frac{1}{2} \neq 0$$

We can conclude that the series is divergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

No conclusion about convergence or divergence can be drawn from this result.

(This is divergence test failure.)

Theorem: Some Properties of Convergent Series

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are convergent series with sums α and β, respectively. Then the series

$$\sum(a_k + b_k), \quad \sum(a_k - b_k), \quad \text{and} \quad \sum c a_k \ \text{ for constant } c$$

are convergent with sums

$$\sum(a_k + b_k) = \alpha + \beta, \quad \sum(a_k - b_k) = \alpha - \beta,$$

$$\text{and} \quad \sum c a_k = c\alpha.$$

Example

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{4}{n(n+1)}+\frac{2}{5^{n-1}}\right)$$

$$\sum_{n=1}^{\infty}\frac{4}{n(n+1)} = \sum_{n=1}^{\infty}4\left(\frac{1}{n(n+1)}\right) = 4(1) = 4$$

$$\sum_{n=1}^{\infty}\frac{2}{5^{n-1}} = \sum_{n=1}^{\infty}2\left(\frac{1}{5}\right)^{n-1} = \frac{2}{1-\frac{1}{5}} = \frac{10}{4} \qquad |r| = \frac{1}{5} < 1$$

So $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n(n+1)}+\frac{2}{5^{n-1}}\right) = 4 + \frac{10}{4} = \frac{13}{2}$

Section 11.3: The Integral Test

Recall: Integrals were defined in terms of sums—Riemann Sums—and there is a geometric way, relating to area between curves, to interpret them.

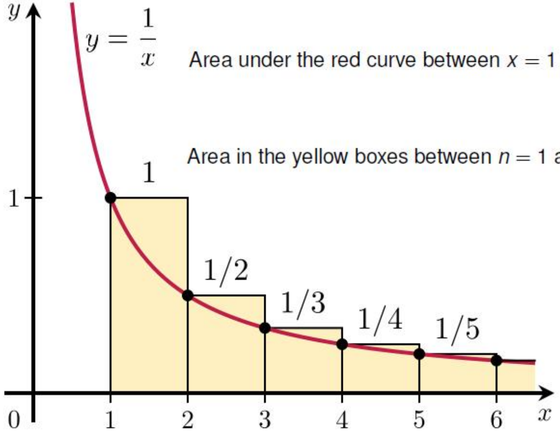
Note: A series can be related to areas too

$$a_1 + a_2 + \cdots = a_1 \cdot 1 + a_2 \cdot 1 + \cdots$$

if the numbers a_k are heights and all the widths are 1. Of course, this makes best sense when the numbers a_k are positive.

Context for this Section: We will restrict our attention <u>for the moment</u> to series of nonnegative terms.

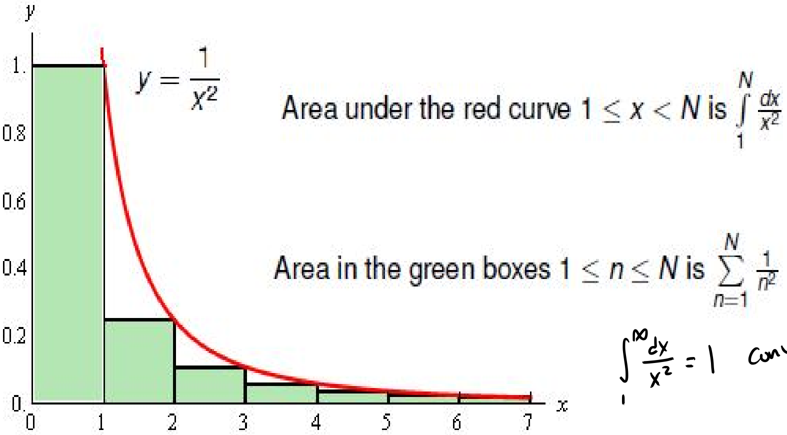
Relating an Integral to a Series (divergent)



$\int_1^\infty \frac{dx}{x}$ diverges. it's infinite.

Figure: Comparison of *areas* related to $\int_1^\infty \frac{dx}{x}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$.

Relating an Integral to a Series (convergent)



$\int_1^\infty \frac{dx}{x^2} = 1$ convergent

Figure: Comparison of *areas* related to $\int_1^\infty \frac{dx}{x^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Set Up for the Integral Test

Question: Does the series of positive terms $\sum_{n=1}^{\infty} a_n$ converge or diverge?

- Suppose f is a continuous, positive, decreasing function defined on the interval $[1, \infty)$.
- Also suppose that $a_n = f(n)$—the function and the terms in the series have the same "formula".
- Assume that we are able to determine if the integral $\int_1^{\infty} f(x)\,dx$ converges or diverges.

Geometric Interpretation of the Integral Test

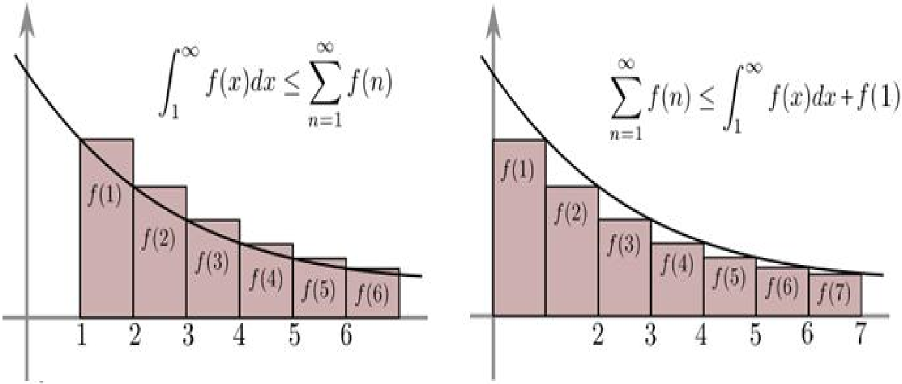


Figure: The possible value of the series can be trapped between the possible values of integrals.

The Integral Test

Theorem: Let $\sum a_n$ be a series of positive terms and let the function f defined on $[1, \infty)$ be continuous, positive and decreasing with

$$a_n = f(n).$$

(i) If $\int\limits_1^\infty f(x)\,dx$ is convergent, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int\limits_1^\infty f(x)\,dx$ is divergent, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Both series and integral converge, or both series and integral diverge.

Examples:

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Integral test: $f(x) = \dfrac{1}{x^2+1}$

$f(n) = \dfrac{1}{n^2+1}$, f is positive, continuous,

and decreasing on $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dt$$

$$= \lim_{t\to\infty} \tan^{-1}x \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}(t) - \tan^{-1}(1)\right) = \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

The integral converges. By the theorem the series also converges.