# April 3 Math 3260 sec. 52 Spring 2024

## Section 5.1: Eigenvectors and Eigenvalues

Consider the matrix $A$ and vectors $\mathbf{u}$ and $\mathbf{v}$.

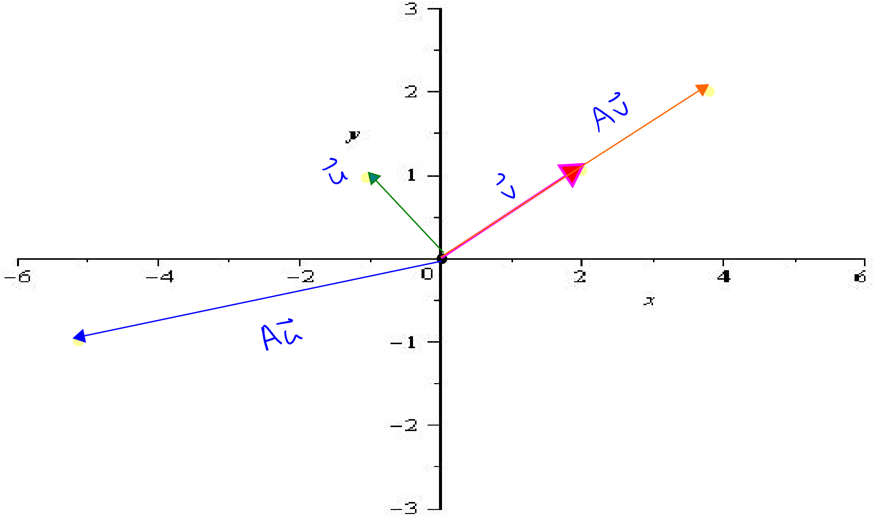
$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Plot $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$ on the axis on the next slide.

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

# Example Plot




Figure

# Eigenvalues and Eigenvectors

**Remark:** Note the action of $A$ on the two vectors seems fundamentally different.

- $A$ seems to both stretch and rotate the vector $\mathbf{u}$.
- The *action of A* on the vector $\mathbf{v}$ is just a stretch/compress.

$$\boxed{A\mathbf{v} \text{ is in } \mathrm{Span}\{\mathbf{v}\}.}$$

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant $\lambda$—and for nonzero vector $\mathbf{x}$.

# Definition of Eigenvector and Eigenvalue

**Definition:**

Let $A$ be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar $\lambda$ is called an **eigenvector** of the matrix $A$.

A scalar $\lambda$ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix $A$. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* $\lambda$.

**Note** that built right into this definition is that the eigenvector $\mathbf{x}$ MUST BE a nonzero vector!

# Example

The number $\lambda = -4$ is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding eigenvectors.

We want vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $A\vec{x} = -4\vec{x}$.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1x_1 + 6x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 \\ -4x_2 \end{bmatrix}$$

$$1x_1 + 6x_2 = -4x_1$$
$$5x_1 + 2x_2 = -4x_2$$

subtract $-4x_1$ or $-4x_2$

$$(1-(-4))x_1 + 6x_2 = 0$$
$$5x_1 + (2-(-4))x_2 = 0$$

homogeneous system

In matrix format this is

$$\begin{bmatrix} 1-(-4) & 6 \\ 5 & 2-(-4) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using an augmented matrix

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_1 = \frac{-6}{5} x_2$$

$x_2$ is free.

The eigenvectors are of the form

$$\vec{x} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}$$ w/ $x_2$ any nonzero number.

The eigenvectors are the nonzero vectors in $\text{Span}\left\{\begin{bmatrix} -6/5 \\ 1 \end{bmatrix}\right\}$.

# Eigenspace

**Definition:**

Let $A$ be an $n \times n$ matrix and $\lambda$ and eigenvalue of $A$. The set of all eigenvectors corresponding to $\lambda$ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of $A$ corresponding to $\lambda$.**

**Remark:** The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

▸ Eigenvalues & Eigenvectors Video

# Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of $A$ corresponding to $\lambda$.

We'll find a basis for the null space of $A - 2I$.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

considering

$$(A - 2I)\vec{x} = \vec{0}$$

$$A - 2I \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x_1 = \frac{1}{2}x_2 - 3x_3$$

$x_2, x_3$ are free.

An eigenvector looks like

$$\vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigenspace is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

# Matrices with Nice Structure

**Theorem:**

If $A$ is an $n \times n$ triangular matrix, then the eigenvalues of $A$ are its diagonal elements.

**Example:** Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & \pi & 0 \\ -1 & 0 & 1 \end{bmatrix}$

The eigen values are

$\lambda_1 = 3, \quad \lambda_2 = \pi, \quad \lambda_3 = 1$