April 4 Math 3260 sec. 51 Spring 2022

Section 4.4: Coordinate Systems

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each vector $\mathbf{x}$ in V, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

- **Remark:** It's clear that each vector can be written as a linear combination because a basis is a spanning set.

- **Remark:** This is saying that it can only be done in one way—that is, there is only one set of numbers $c_1, \ldots, c_n$.

Uniqueness of Coefficients

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V. If

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots c_n\mathbf{b}_n \quad \text{and} \\ \mathbf{x} &= a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots a_n\mathbf{b}_n, \end{aligned}$$

show that $a_1 = c_1, a_2 = c_2, \ldots, a_n = c_n$.

We can create a homogeneous equation by subtracting the bottom line from the top.

$$\vec{0} = (c_1 - a_1)\vec{b}_1 + (c_2 - a_2)\vec{b}_2 + \cdots + (c_n - a_n)\vec{b}_n$$

Because the basis is linearly independent the coefficients must all be zero.

That is $c_1 - a_1 = 0 \Rightarrow a_1 = c_1$

$c_2 - a_2 = 0 \Rightarrow a_2 = c_2$

etc.

Consequence of Linear Independence

Take the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is true that $\mathbb{R}^2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Consider $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Note that we can write $\mathbf{x}$ in two different ways

$$\begin{aligned} \mathbf{x} &= 2\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3 \quad \text{and} \\ \mathbf{x} &= 1\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3. \end{aligned}$$

Why doesn't this contradict our theorem?

The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not linearly independent.

Coordinate Vectors

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

Example

Let $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) in $\mathbb{P}_3$. Determine $[\mathbf{p}]_{\mathcal{B}}$ where

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$ $= 3(1) + 0(t) + (-4)(t^2) + 6t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}$$

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Example

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

We need c_1 and c_2 such that

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

or as a matrix equation

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\vec{b}}$$

we can use Cramer's rule

Call the coefficient matrix A.

$$\det\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = 2+1 = 3, \quad A_1(\vec{b}) = \begin{bmatrix} 4 & -1 \\ 5 & 1 \end{bmatrix} \quad \det(A_1(\vec{b})) = 9$$

$$A_2(\vec{b}) = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} \qquad \det(A_2(\vec{b})) = 6$$

$$c_1 = \frac{\det(A_1)}{\det(A)} = \frac{9}{3} = 3, \qquad c_2 = \frac{\det(A_2)}{\det(A)} = \frac{6}{3} = 2$$

Hence

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Coordinates in $\mathbb{R}^n$

Note from this example that $\mathbf{x} = P_\mathcal{B}[\mathbf{x}]_\mathcal{B}$ where $P_\mathcal{B}$ is the matrix $[\mathbf{b}_1\ \mathbf{b}_2]$. The matrix $P_\mathcal{B}$ is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ is the linear transformation defined by

$$[\mathbf{x}]_\mathcal{B} = P_\mathcal{B}^{-1}\mathbf{x}$$

where the matrix

$$P_\mathcal{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Example

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Determine the matrix $P_{\mathcal{B}}$ and its inverse.

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad P_{\mathcal{B}}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\det(P_{\mathcal{B}}) = 3$$

Use this to find

(a) the coordinate vector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) a vector **x** whose coordinate vector is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

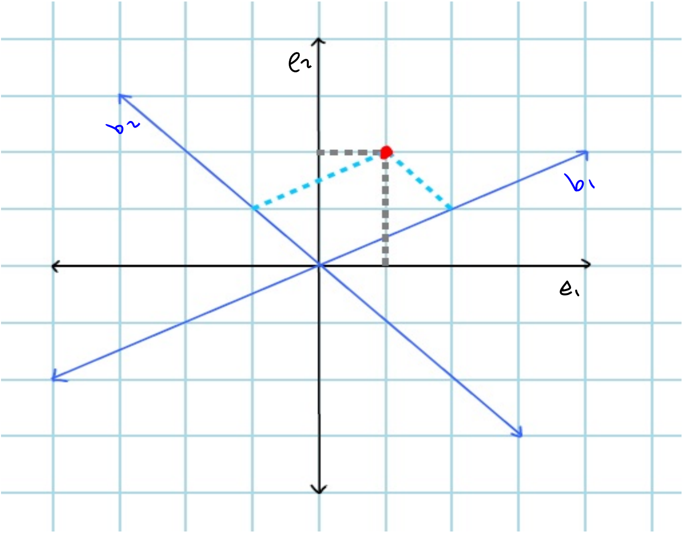

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.

Theorem: Coordinate Mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark: When such a mapping exists, we say that V is **isomorphic** to $\mathbb{R}^n$. Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

$\mathbb{P}_3$ is **Isomorphic** to $\mathbb{R}^4$

We saw that using the ordered basis $\mathcal{B} = \{1, t, t^2, t^3\}$ that any vector

$$\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

in $\mathbb{P}_3$ has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

in $\mathbb{R}^4$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

Let's use the basis $\mathcal{B} = \{1, t, t^2\}$.

The coordinate vectors are

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [\vec{r}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\{\vec{p}, \vec{q}, \vec{r}\}$ is linearly independent in $\mathbb{P}_2$ if and only if $\{[\vec{p}]_{\mathcal{B}}, [\vec{q}]_{\mathcal{B}}, [\vec{r}]_{\mathcal{B}}\}$ is lin. indep.

in $\mathbb{R}^3$. Is there a set of coefficients c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

We can use a matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

We can use the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ -2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The last column is a pivot column, hence the system is inconsistent.

The vectors in $\mathbb{R}^3$ are lin. independent, so $\{\vec{p}, \vec{q}, \vec{r}\}$ are lin. independent in $\mathbb{P}_2$.