April 5 Math 2306 sec. 51 Spring 2023

Section 13: The Laplace Transform

Recall that we defined the Laplace transform of a function defined on $[0, \infty)$.

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

We can compute a Laplace transform using this definition. More often, we will use a table of Laplace transforms.

A Limited Table of Laplace Transforms

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},$

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{4 - 4t + t^2\}$$

$$= 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1!}{s^2}\right) + \frac{2!}{s^3}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(d) $f(t) = \sin^2 5t$ $\qquad \mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$,

How would we evaluate $\int \sin^2 5t \, dt$

$$\sin^2 5t = \frac{1}{2} - \frac{1}{2}\cos(10t)$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s}{s^2+10^2}$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2}\,\frac{s}{s^2+100}$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

f grows at most like an exponential as $t \to \infty$

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c>0$, then f has a Laplace transform for $s>c$.

An example of a function that is NOT of exponential order for any c is $f(t)=e^{t^2}$. Note that

$$f(t)=e^{t^2}=\left(e^t\right)^t \quad\Longrightarrow\quad |f(t)|>e^{ct} \quad \text{whenever} \quad t>c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Using a Table

When using a table of Laplace transforms, the expression must match exactly. For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

so

$$\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3.$$

Note that $n = 3$, so there must be 3! in the numerator and the power $4 = 3 + 1$ on s.

What is $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$?

Find the Inverse Laplace Transform

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

If $n+1=7$, $n=6$ we need $6!$ on top.

$$\frac{1}{s^7} = \frac{6!}{6!}\,\frac{1}{s^7} = \frac{1}{6!}\,\frac{6!}{s^7}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}\,t^6$$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

$$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\}$$

$k^2 = 9$

$k = 3$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{3}{3}\,\frac{1}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

we need partial fractions

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \qquad s(s-2)$$

$$s-8 = A(s-2) + Bs$$

Set $s=0$ $\quad 0-8 = A(0-2) + B(0)$

$$-8 = -2A \Rightarrow A = 4$$

set $s=2$ $\quad 2-8 = A(2-2) + B(2)$

$$-6 = 2B \Rightarrow B = -3$$

$$\frac{s-8}{s(s-2)} = \frac{4}{s} - \frac{3}{s-2}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4(1) - 3(e^{2t})$$

$$= 4 - 3e^{2t}$$