April 5 Math 3260 sec. 51 Spring 2024

Section 5.1: Eigenvectors and Eigenvalues

Definition:

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Eigenspace

Definition:

Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\}, = \text{Nul}(A - \lambda I)$$

is called the **eigenspace of A corresponding to λ.**

Remark: The video mentioned something called an **Eigenbasis**. When possible, an **eigenbasis** will be constructed by taking bases for all eigenspaces for a matrix and combining them.

We'll get back to this in section 5.3 when we talk about *diagonalizability*.

Matrices with Nice Structure

Theorem:

If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & -8 & 0 \\ 1 & 2 & 3 & 489 \end{bmatrix}$

$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -8$

and $\lambda_4 = 489$.

Example

Suppose $\lambda = 0$ is an eigenvalue[1] of a matrix A. Argue that A is not invertible.

If $\lambda = 0$ is an eigenvalue, then there is a nonzero vector $\vec{x}$ such that

$$A\vec{x} = 0\vec{x} \quad \text{i.e.,} \quad A\vec{x} = \vec{0}.$$

Since the homogeneous equation has a nontrivial solution, A can't be invertible

[1] Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!

Theorems

Theorem:

A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem:

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Linear Independence

Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix A corresponding to distinct eigenvalues λ_1 and λ_2 (i.e.$\lambda_1 \neq \lambda_2$).

Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

We know, $\vec{v}_1 \neq 0$, $\vec{v}_2 \neq 0$, $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$. Also $\lambda_1 \neq \lambda_2$. Let's consider the homogeneous equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \quad (*)$$

we need to show that $c_1 = 0$ and $c_2 = 0$ (this is the only solution.

We'll create two equations from $(*)$.

① Multiply (*) by A

② Multiply (*) by λ_1

① $A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(\vec{0})$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0}$$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$$ ← save this.

② $\lambda_1(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_1(\vec{0})$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}$$ ← save this.

Subtract the bottom eqn from the top

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$$

$$-\quad c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}$$

$$\vec{0} + c_2\lambda_2\vec{v}_2 - c_2\lambda_1\vec{v}_2 = \vec{0}$$

$$c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0}$$

so $c_2 = 0$, or $\lambda_2 - \lambda_1 = 0$, or $\vec{v}_2 = \vec{0}$.

But $\vec{v}_2 \neq \vec{0}$ and $\lambda_2 - \lambda_1 \neq 0$, so it must be that $c_2 = 0$. So equation (*) becomes

$$c_1\vec{v}_1 + 0\vec{v}_2 = \vec{0} \implies c_1\vec{v}_1 = \vec{0}.$$

Since $\vec{v}_1 \neq \vec{0}$, it must also be that $c_1 = 0$. Hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

Another Addendum to the Invertible Matrix Thm.

Theorem:

The $n \times n$ matrix A is invertible if and only if[a]

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is nonzero.

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[a] This is nothing new, we're just adding to the list.

Section 5.2: The Characteristic Equation

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ by appealing to the fact that the equation $A\mathbf{x} = \lambda I_2\mathbf{x}$ can be restated as:

Find a nontrivial solution of the homogeneous equation

$$(A - \lambda I_2)\mathbf{x} = \mathbf{0}.$$

we need $A - \lambda I$ to be singular. This requires $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 3\cdot 3$$

$$= \lambda^2 + 4\lambda - 12 - 9$$

$$= \lambda^2 + 4\lambda - 21$$

we need

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda + 7)(\lambda - 3) = 0$$

we get two eigenvalues,

$$\lambda_1 = -7, \quad \lambda_2 = 3.$$

Characteristic Equation

Definition:

For $n \times n$ matrix A, the expression $\det(A - \lambda I)$ is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition:

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Theorem:

The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.