April 6 Math 3260 sec. 52 Spring 2022

Section 4.5: Dimension of a Vector Space

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

We saw an example of this in $\mathbb{R}^n$. Remember that if we had more vectors than entries in each vector, the set was automatically linearly dependent. For example

$$\left\{ \begin{bmatrix} 1\\2\\3 \end{bmatrix}, \begin{bmatrix} 2\\10\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\4 \end{bmatrix}, \begin{bmatrix} 3\\6\\8 \end{bmatrix} \right\}$$

is linearly dependent because there are 4 vectors from $\mathbb{R}^3$.

Example

Recall that a basis for $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$. Is the set

$$\{1 + t, 2t - 3t^3, 1 + t + t^2, 1 + t + t^2 + t^3, 2 - t + 2t^3\}$$

linearly dependent or linearly independent?

This is 5 vectors. $\mathbb{P}_3$ has a basis with 4 vectors in it. Since this set has more than 4 vectors, it is linearly dependent.

Dimension

Corollary: If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Definition: If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[1], then V is said to be **infinite dimensional**.

[1] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Find $\dim(\mathbb{R}^n)$.

A basis for $\mathbb{R}^n$ is $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ It has n vectors.

$$\dim(\mathbb{R}^n) = n$$

(b) Determine dim Col A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

A is an ref.

A has 2 pivot columns so a basis for Col(A) would have 2 vectors in it.

$$\dim(\text{Col}(A)) = 2.$$

Some Geometry in $\mathbb{R}^3$

We can describe all of the subspaces of $\mathbb{R}^3$ geometrically. The subspace(s) of dimension

(a) zero: is just the origin (one point), $(0,0,0)$.

(b) one: are lines through the origin. $\mathsf{Span}\{\mathbf{u}\}$ where $\mathbf{u}$ is not the zero vector.

(c) two: are planes that contain the origin and two other, noncolinear points. $\mathsf{Span}\{\mathbf{u},\mathbf{v}\}$ with $\{\mathbf{u},\mathbf{v}\}$ linearly independent.

(d) three: is all of $\mathbb{R}^3$.

Subspaces and Dimension

Theorem: Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Subspaces and Dimension

Theorem: Let V be a vector space with $\dim V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Remark: this connects two properties **spanning** and **linear independence**. If $\dim V = p$ and a set contains p vectors then

- linear independence $\Longrightarrow$ spanning
- spanning $\Longrightarrow$ linear independence

Again, this is **IF** the number of vectors matches the dimension of the vector space.

Column and Null Spaces

Theorem: Let A be an $m \times n$ matrix. Then

$$\dim \mathrm{Nul} A \;=\; \text{the number of free variables in the equation } \; A\mathbf{x} = \mathbf{0},$$

and

$$\dim \mathrm{Col} A \;=\; \text{the number of pivot positions in } \; A.$$

Example

A matrix A is show along with its rref. Find the dimensions of the null space and column space of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $A\vec{x} = \vec{0}$, there are 1 free variables because there is one non pivot column.

$\dim(\text{Nul}(A)) = 1$

There are 3 pivot columns, hence

$\dim(\text{Col}(A)) = 3$

Example

A matrix A along with its rref is shown.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Row A and state the dimension dim Row A.

To get a basis for Row(A), we use the non zero rows from the rref. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$$

$$\dim(\text{Row}(A)) = 3$$

Example continued ...

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find a basis for Col A and state its dimension.

The basis vectors for Col(A) come from A.

A basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$.

$\dim(\mathrm{Col}(A)) = 3$

Example continued ...

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Find a basis for Nul A and state its dimension.

To get a basis, we need to characterize solutions to $A\vec{x}=\vec{0}$. For $\vec{x}$ in Nul(A)

$$\vec{x} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

A basis for Nul A is

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

$$\dim(\text{Nul}A) = 2.$$

Remarks

- Row operations preserve row space, but change linear dependence relations of rows. Row operations change column space, but preserve linear dependence relations of columns.

- Another way to obtain a basis for Row A is to take the transpose A^T and do row operations. We have the following relationships:

$$\text{Col } A = \text{Row } A^T \quad \text{and} \quad \text{Row } A = \text{Col } A^T.$$

Rank & Nullity

Definition: The **rank** of a matrix A, denoted $\mathrm{rank}(A)$, is the dimension of the column space of A.

Definition: The **nullity** of a matrix A is the dimension of the null space.

Remark: Since the dimension of the column space is the number of pivot positions, $\dim \mathrm{Col}(A) = \dim \mathrm{Row}(A)$. That is, the rank, dimension of the columns space, and dimension of the row space are all the same.

The Rank-Nullity Theorem

Theorem: For $m \times n$ matrix A, dim Col A = dim Row A = rank A. Moreover

$$\text{rank } A + \text{dim Nul } A = n.$$

Note: This theorem states the rather obvious fact that

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{non-pivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{total number} \\ \text{of columns} \end{array} \right\}.$$

Examples

(1) A is a 5×4 matrix with rank $A = 4$. What is dim Nul A?

rank + nullity = n

Here $n = 4$ and $\text{rank}(A) = 4$

$4 + \dim(\text{Nul } A) = 4$

$\Rightarrow \dim(\text{Nul } A) = 0$

Examples

rank + nullity = n

(2) If A is 7×5 and dim Col $A = 2$. Determine

1. the nullity of A Here, rank$(A) = 2$ and $n = 5$

   the nullity $= 5 - 2 = 3$

2. the rank of A^T

   rank$(A^T) = \dim(\text{Col } A^T) = \dim(\text{Row } A^T) = \dim(\text{Col } A) = 2$

3. the nullity of A^T

   A^T is 5×7 so the $n = 7$ for A^T

   nullity of $A^T = 7 - 2 = 5$.

Addendum to Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following are equivalent to the statement that A is invertible.

(m) The columns of A form a basis for $\mathbb{R}^n$

(n) $\text{Col } A = \mathbb{R}^n$

(o) $\dim \text{Col } A = n$

(p) $\text{rank } A = n$

(q) $\text{Nul } A = \{\mathbf{0}\}$

(r) $\dim \text{Nul } A = 0$

Section 5.1: Eigenvectors and Eigenvalues[2]

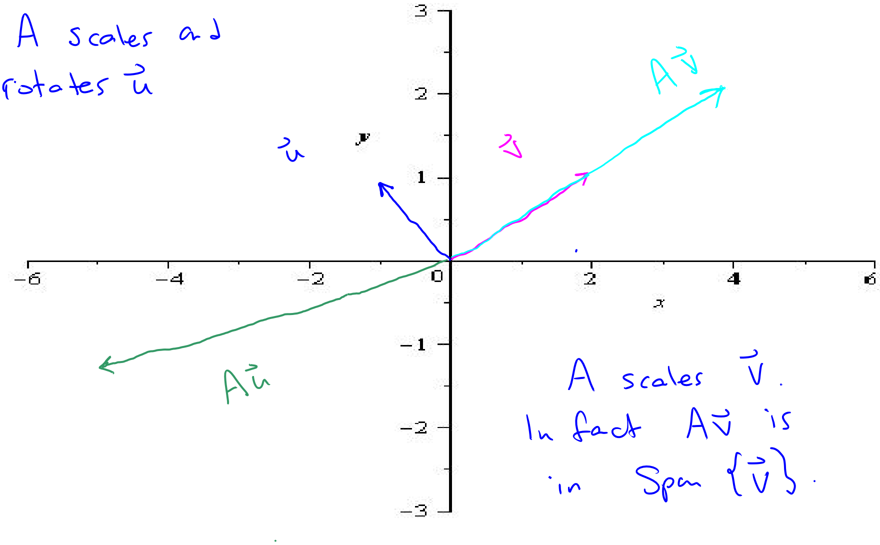
Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Plot $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$ on the axis on the next slide.

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

[2] We'll forgo section 4.6 and come back if time allows

Example Plot



Eigenvalues and Eigenvectors

Note that in this example, the matrix A seems to both stretch and rotate the vector $\mathbf{u}$. But the *action of A* on the vector $\mathbf{v}$ is just a stretch/compress. That is $A\mathbf{v}$ is in $\mathrm{Span}\{\mathbf{v}\}$.

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant λ—and for nonzero vector $\mathbf{x}$.