April 8 Math 3260 sec. 51 Spring 2024

Section 5.2: The Characteristic Equation

Definition:

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Eigenspace

Definition:

Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\} = \mathrm{Nul}(A - \lambda I)$$

is called the **eigenspace of A corresponding to** λ.

Finding Eigenvalues

The requirement that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has **non-trivial** solutions can be restated as the condition

$$\det(A - \lambda I) = 0.$$

This is a scalar equation for the number(s) λ.

Characteristic Equation

Definition:

For $n \times n$ matrix A, the expression $\det(A - \lambda I)$ is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition:

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Theorem:

The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we need $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

Characteristic eqn is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

$$x^4 - 14x^3 + 68x^2 - 130x + 75 = 0$$

The eigenvalues are

$$\lambda_1 = 5, \quad \lambda_2 = 3, \quad \text{and} \quad \lambda_3 = 1$$

Multiplicities

Definition:

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

$\lambda = 5$ has algebraic multiplicity 2

Since $\lambda - 5$ is a double factor.

To find the geometric multiplicity, we need to find a basis for the eigenspace

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Look for solutions to $(A - 5I)\vec{x} = \vec{0}$.

$$[A \ \vec{0}] = \begin{bmatrix} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & -8 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 - is free

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

Eigenvectors look like

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with one free variable the eigenspace for $\lambda = 5$ has dimension 1.

The geometric multiplicity of $\lambda = 5$ is one.

Similarity

Definition:

Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[a].

[a] **Note:** similarity is NOT related to row equivalence.

Theorem:

If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

$$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$$

Note
$I = P^{-1} I P$

$$= \det(P^{-1}AP - \lambda P^{-1} I P)$$

$$= \det(P^{-1}(AP - \lambda I P))$$

$$= \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

$$= \det(A - \lambda I) \det(P^{-1}) \det(P)$$

$$= \det(A - \lambda I)\, \det(P^{-1}P)$$

$$= \det(A - \lambda I) \cdot 1$$

$$= \det(A - \lambda I)$$