April 8 Math 3260 sec. 52 Spring 2022

Section 5.1: Eigenvectors and Eigenvalues

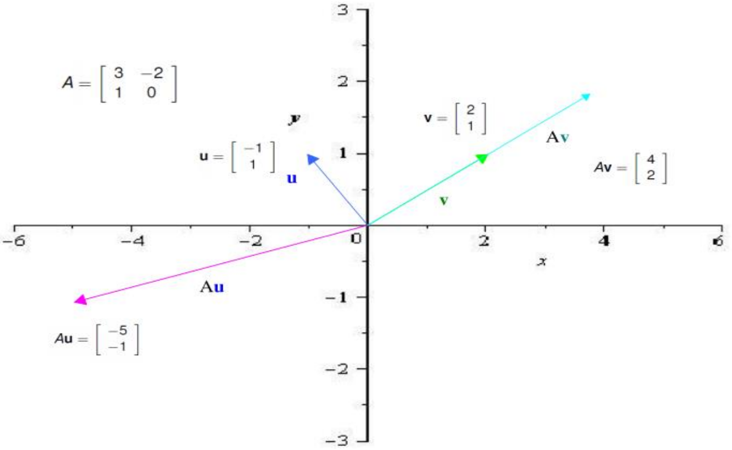

Figure: We saw that multiplication by the matrix A scaled and rotated $\mathbf{u}$. It only scaled $\mathbf{v}$. In fact, $A\mathbf{v} = 2\mathbf{v}$.

Eigenvalues and Eigenvectors

Most vectors are expected to be like $\mathbf{u}$, without any obvious relationship between $\mathbf{u}$ and $A\mathbf{u}$. The relationship between $\mathbf{v}$ and $A\mathbf{v}$ is remarkable in that $A\mathbf{v}$ is contained in $\text{Span}\{\mathbf{v}\}$.

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant λ—and for nonzero vector $\mathbf{x}$.

Definition of Eigenvector and Eigenvalue

Definition: Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Note that built right into this definition is that the eigenvector **x must be** nonzero!

Example

The number $\lambda = -4$ is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding eigenvectors.

We want to find nonzero vector(s) $\vec{x}$ such that

$A\vec{x} = \lambda\vec{x}$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 6x_2 \\ 5x_1 + 2x_2 \end{bmatrix} = -4\vec{x} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 \\ -4x_2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_1 + 6x_2 &= -4x_1 \\ 5x_1 + 2x_2 &= -4x_2 \end{aligned}$$

We can subtract $-4x_1$ or $-4x_2$ from each equation.

$$(1-(-4))x_1 + 6x_2 = 0$$

homogeneous system

$$5x_1 + (2-(-4))x_2 = 0$$

$$\begin{aligned} 5x_1 + 6x_2 &= 0 \\ 5x_1 + 6x_2 &= 0 \end{aligned} \implies \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Using an augmented matrix

$$\left[\begin{array}{cc:c} 5 & 6 & 0 \\ 5 & 6 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{-6}{5} x_2$

x_2 - free

So the solutions

$$\vec{x} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}.$$

The eigenvectors are all

$$\vec{x} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix} \text{ for } x_2 \neq 0.$$

We can check this for some choice of $\vec{x}$. If $x_2 = 5$, then $\vec{x} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \end{bmatrix} = -4 \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of A corresponding to λ.

We're finding a basis for $\text{Nul}(A - \lambda I)$.

$$A - \lambda I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_2 - 3x_3$

x_2, x_3 are free

Solutions to $(A - \lambda I)\vec{x} = \vec{0}$ look like

$$\vec{x} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigen space is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Matrices with Nice Structure

Theorem: If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & \pi & 0 \\ -1 & 0 & 1 \end{bmatrix}$

A is lower, triangular with eigen values

$3, \pi,$ and 1.

Example

Suppose $\lambda = 0$ is an eigenvalue[1] of a matrix A. Argue that A is not invertible.

We know there is a nonzero vector $\vec{x}$

such that $A\vec{x} = 0\vec{x} = \vec{0}$.

That is $A\vec{x} = \vec{0}$ has a nontrivial

solution.

Hence A is singular.

[1] Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!

Theorems

Theorem: A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.