August 12 Math 2306 sec. 51 Fall 2024

Section 1: Concepts and Terminology

Our first goal is to define what a *differential equation* is and identify relevant properties and the language used to talk about this subject.

Let's look at a simple example of how an equation might arise based on assumptions and modeling how something works. Consider the question:

Why are most cells microscopic?

Modeling Cell Growth

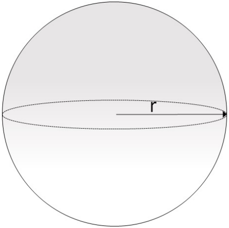

Figure: An idealized cell as a sphere of radius r.

We'll make the following assumptions:

- The density ρ of the cell is constant, so the mass $m = \rho V$ (where V is the cell volume).
- Nutrient intake is proportional to the cell's surface area
- Metabolic expenditure is proportional to cell volume.

A balance equation

The basic idea is called a **balance** equation:

$$\begin{pmatrix}\text{rate of change}\\ \text{of mass}\end{pmatrix} = \begin{pmatrix}\text{Uptake rate}\\ \text{of nutrients}\end{pmatrix} - \begin{pmatrix}\text{Metabolic use rate}\\ \text{of nutrients}\end{pmatrix}$$

$m = \rho V$

αS

surface area

βV

α, β are constants of proportionality

$$\frac{dm}{dt} = \alpha S - \beta V$$

we can rewrite this in terms of r.

$$S = 4\pi r^2 \qquad V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}\left(\rho \frac{4}{3}\pi r^3\right) = \alpha 4\pi r^2 - \beta \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}\left(\rho \frac{4}{3}\pi r^3\right) = \rho \frac{4}{3}\pi \frac{d}{dt}\left(r^3\right)$$

$$= \rho \frac{4}{3}\pi \left(3r^2\right)\frac{dr}{dt}$$

$$\rho \frac{4}{3}\pi \left(3r^2\right)\frac{dr}{dt} = 4\pi \alpha r^2 - \frac{4}{3}\pi \beta r^3$$

$$4\pi \rho r^2 \frac{dr}{dt} = 4\pi \alpha r^2 - \frac{4}{3}\pi \beta r^3$$

$$\rho r^2 \frac{dr}{dt} = \alpha r^2 - \frac{1}{3}\beta r^3$$

for $r \neq 0$, cancel r^2

$$\rho \frac{dr}{dt} = \alpha - \frac{1}{3}\beta r$$

This equation connects $\frac{dr}{dt}$ and r.

$$\frac{dr}{dt} = \frac{1}{\rho}\left(\alpha - \frac{1}{3}\beta r\right)$$

$$\frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right).$$

Side note $\quad \int \frac{dy}{dt}\, dt = y(t) + C$

Can we find r from

$$\int \frac{dr}{dt}\,dt = \int \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right)dt \quad ?$$

we don't know what r(t) is.

We can't determine what integration results might be needed without knowing what kind of function r is.

Analysis of the Radius Equation

$$\frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right).$$

Note that $\frac{\beta}{3\rho}$ and $\frac{3\alpha}{\beta}$ are some positive numbers. We can use some basic calculus ideas to glean information about cell size.

Recall if $f'(t) > 0$, f is increasing

if $f'(t) < 0$, f is decreasing.

Note: if $\frac{3\alpha}{\beta} - r > 0 \Rightarrow r < \frac{3\alpha}{\beta}$

then $\frac{dr}{dt} > 0$, r grows

If $\frac{3\alpha}{\beta} - r < 0$, $r > \frac{3\alpha}{\beta}$

$\frac{dr}{dt} < 0$, $\therefore$ r shrinks

Without even trying to "Solve" the equation, we see that there's a limit on how r can grow. We even see that the actual limit is the number $\frac{3\alpha}{\beta}$.

r can't get bigger than this number because it will start to decrease as soon as it does.