August 12 Math 2306 sec. 53 Fall 2024

Section 1: Concepts and Terminology

Our first goal is to define what a *differential equation* is and identify relevant properties and the language used to talk about this subject.

Let's look at a simple example of how an equation might arise based on assumptions and modeling how something works. Consider the question:

Why are most cells microscopic?

Modeling Cell Growth

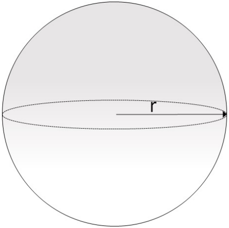

Figure: An idealized cell as a sphere of radius r.

We'll make the following assumptions:

- The density ρ of the cell is constant, so the mass $m = \rho V$ (where V is the cell volume).
- Nutrient intake is proportional to the cell's surface area
- Metabolic expenditure is proportional to cell volume.

A balance equation

The basic idea is called a **balance** equation:

$$\begin{pmatrix} \text{rate of change} \\ \text{of mass} \end{pmatrix} = \begin{pmatrix} \text{Uptake rate} \\ \text{of nutrients} \end{pmatrix} - \begin{pmatrix} \text{Metabolic use rate} \\ \text{of nutrients} \end{pmatrix}$$

$m = \rho V$ $\qquad\qquad$ $\uparrow$ $\qquad\qquad$ $\uparrow$

Change rate $\qquad$ αS $\qquad\qquad$ βV

$\frac{dm}{dt}$

α, β are constants of proportionality.

$$\frac{dm}{dt} = \alpha S - \beta V$$

$$S = 4\pi r^2 , \qquad V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}\left(\rho \frac{4}{3}\pi r^3\right) = \alpha 4\pi r^2 - \beta \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}\left(\rho \frac{4}{3}\pi r^3\right) = \rho \frac{4}{3}\pi \frac{d}{dt}\left(r^3\right)$$

$$= \rho \frac{4}{\cancel{3}}\pi \left(\cancel{3} r^2\right) \frac{dr}{dt}$$

$$\rho 4\pi r^2 \frac{dr}{dt} = 4\pi \alpha r^2 - \frac{4}{3}\pi \beta r^3$$

Cancel 4π

$$\rho r^2 \frac{dr}{dt} = \alpha r^2 - \frac{1}{3}\beta r^3$$

for $r \neq 0$, cancel r^2

$$\rho \frac{dr}{dt} = \alpha - \frac{1}{3}\beta r$$

Divide by ρ

$$\frac{dr}{dt} = \frac{1}{\rho}\left(\alpha - \frac{1}{3}\beta r\right)$$

$$\frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right).$$

This equation shows how $\frac{dr}{dt}$ is related to r.

Note: $\int \frac{dy}{dt}\, dt = y(t) + C$

Could we find $r(t)$

$$\int \frac{dr}{dt}\, dt = \int \frac{\beta}{3\rho}\left(\frac{3q}{\beta} - r\right) dt$$

we don't know what kind of function r is.

Analysis of the Radius Equation

$$\frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right).$$

Note that $\frac{\beta}{3\rho}$ and $\frac{3\alpha}{\beta}$ are some positive numbers. We can use some basic calculus ideas to glean information about cell size.

Recall: If $f'(t) > 0$, f is increasing

$f'(t) < 0$, f is decreasing

If $\frac{3\alpha}{\beta} - r > 0$, i.e. $r < \frac{3\alpha}{\beta}$

$\frac{dr}{dt} > 0$, r increases (cell growth)

If $\frac{3\alpha}{\beta} - r < 0$, $r > \frac{3\alpha}{\beta}$

$\frac{dr}{dt} < 0$, r shrinks

The equation itself indicates that there is a limit on the cell size. It even gives the specific number $\frac{3\alpha}{\beta}$ as the maximum radius.