

August 14 Math 2306 sec. 53 Fall 2024

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $\frac{dy}{dx} = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2 \sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4 \cos(2x)$. Note that this function happens to satisfy the equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

$$y'' = -4 \cos(2x) \quad , \quad 4y = 4 \cos(2x)$$

Equations like

$$\frac{dr}{dt} = \frac{\beta}{3\rho} \left(\frac{3\alpha}{\beta} - r \right) \quad \text{and} \quad \frac{d^2y}{dx^2} + 4y = 0$$

are examples of differential equations.

Some questions we can ask about this second example are

Questions

1. If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it?
2. Also, is $\cos(2x)$ the only possible function that y could be?

We'll be able to answer these and many more questions as we proceed.

Definition

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A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more independent variables.

We can characterize the *variables* in calculus terms:

Independent and Dependent Variables

An **Independent Variable**: will appear as one that derivatives are taken **with respect to**. (The x in $y = f(x)$.)

A **Dependent^a Variable**: will appear as one that derivatives are taken **of**. (The y in $y = f(x)$.)

^aThink of dependent variables as the functions.

Independent and Dependent Variables

Identify the independent and dependent variables in each differential equation shown.

$$\frac{d^2y}{dx^2} + 4y = 0$$

Independent x Dependent y

$$t^2y'' - 2ty' + y = \sin(t)$$

Independent t Dependent y

$$2\frac{d^2x}{du^2} = x^2 + u^2$$

Independent u Dependent x

Sometimes the independent variable isn't explicitly stated, e.g., consider $y'' + 4y = 0$. I'll usually just call it x or maybe t .

Classifications: Type ODE or PDE

ODEs

An **ordinary differential equation (ODE)** has exactly one independent variable^a. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

^aThese are the subject of this course.

PDEs

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

Classifications: Order

Definition

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

Examples: Identify the order of each differential equation.

$$\frac{dy}{dx} - y^2 = 3x$$

Order = 1st ODE

$$y''' + (y')^4 = x^3$$

Order = 3rd ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

Order = 2nd PDE

Notations and Symbols

We'll mostly use standard derivative notations:

$$\text{Leibniz: } \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \dots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts: } y', \quad y'', \quad \dots \quad y^{(n)}.$$

Newton's **dot notation** is a special notation that is reserved for derivatives **with respect to time**. For example, if $s(t)$ is the position of a moving particle at the time t , then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Note that the dot is like a prime, but it's placed on top of the variable.

Notations and Symbols

On occasion, we'll want to reference a generic ODE. We have a couple of formats for that.

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Example: Express the equation $y'' + 4y = 0$ in the form

$F(x, y, y', \dots, y^{(n)}) = 0$ $y'' + 4y = 0$ is in this
form with $F(x, y, y', y'') = y'' + 4y$

Notations and Symbols

Normal Form

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

E.g. if $n = 1$ $\frac{dy}{dx} = f(x, y)$, if $n = 2$ $\frac{d^2 y}{dx^2} = f(x, y, y')$.

Example: Express the equation $y'' + 4y = 0$ in the normal form.

Subtract $4y$, $y'' = -4y$

$$f(x, y, y') = -4y$$

Notations and Symbols

Differential Form

If $M(x, y)$ and $N(x, y)$ are functions of the variables x and y , then an expression of the form

$$M(x, y) dx + N(x, y) dy$$

is called a **Differential Form**. A first order equation may be written in terms of a differential form as follows:

$$M(x, y) dx + N(x, y) dy = 0$$

Note that this can be rearranged into a couple¹ of different normal forms

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \quad \text{or} \quad \frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}$$

¹We have to assume that N or M is nonzero as needed.

Classifications: Linearity

Linear Differential Equation

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

Example First Order:

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Example Second Order:

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Properties of a Linear ODE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

Properties of Linear ODEs

- ▶ Each of the coefficients a_0, \dots, a_n and the right hand side g may depend on the independent variable but not the dependent one.
- ▶ y , and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- ▶ The characteristic structure of the left side is that

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \quad \cdots, \quad \frac{d^n y}{dx^n}$$

are multiplied by functions of the independent variable and added together.

Examples (Linear -vs- Nonlinear)

Verify that the two equations here are **linear**.

$$y'' + 4y = 0$$

$$t^2 \frac{d^2 x}{dt^2} + 2t \frac{dx}{dt} - x = e^t$$

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$g(x) = 0$$

$$a_0(x) = 4$$

$$a_1(x) = 0$$

$$a_2(x) = 1$$

$$g(t) = e^t$$

$$a_0(t) = -1$$

$$a_1(t) = 2t$$

$$a_2(t) = t^2$$

Examples (Linear -vs- Nonlinear)

Determine why the following two equations are **nonlinear**.

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$y''' + (y')^3 y' = x^3$$

$(y')^4$ is a
nonlinear term

$$u'' + u' = \cos u$$

u is dependent

$\cos u$ is a
nonlinear term

* note: $u'' + u' = \cos(x)$

would
be linear.