

August 16 Math 2306 sec. 51 Fall 2024

Section 1: Concepts and Terminology

We have

- ▶ defined what a differential equation is (an equation involving derivatives),
- ▶ differentiated (pun intended) between independent and dependent variables and between ordinary (ODE) and partial (PDE) differential equations,
- ▶ defined **order** of a differential equation, and
- ▶ defined what it means for an ODE to be **linear**

Recall: A generic n^{th} order ODE is an equation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0$$

or

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

An ODE is said to be **linear** if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

Remark: The characters used for the variables can be changed—they don't have to be x and y .

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(c) $\theta \frac{d^2 u}{d\theta^2} + u \frac{du}{d\theta} = 0$

$u \frac{du}{d\theta}$ is a nonlinear term

- ▶ independent var. θ
- ▶ dependent var. u
- ▶ order 2^{nd}
- ▶ linear/nonlinear nonlinear

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

$$(d) \quad \frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) - 6y = 0$$

Expand

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 6y = 0$$

- ▶ independent var. x
- ▶ dependent var. y
- ▶ order 2nd
- ▶ linear/nonlinear linear

Solutions

Consider the ODE $F(x, y, y', \dots, y^{(n)}) = 0$ (1)

Solutions come in two basic flavors. Let's define what we mean by *solution*.

Definition: Solution (*Explicit Solution*)

A function ϕ defined on an interval I^a and possessing at least n continuous derivatives on I is a **solution** of (1) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

^aThe interval is called the *domain of the solution* or the *interval of definition*.

Solution means what you'd expect. A solution is a function that satisfies the differential equation, i.e., makes it true.

Solution Example

Verify that for any choice of constants c_1 and c_2 , $y = c_1x + \frac{c_2}{x}$ is a solution of the differential equation on the interval $(0, \infty)$.

$$x^2y'' + xy' - y = 0$$

$y = c_1x + \frac{c_2}{x}$ has derivatives of all orders on $I = (0, \infty)$. We'll substitute y into the ODE. We need y' and y'' .

$$y = c_1x + c_2x^{-1}$$

$$y' = c_1 - c_2x^{-2}$$

$$y'' = 0 + 2c_2x^{-3}$$

$$x^2y'' + xy' - y = 0$$

$$x^2(2c_2x^{-3}) + x(c_1 - c_2x^{-2}) - (c_1x + c_2x^{-1}) \stackrel{?}{=} 0$$

$$\underline{2c_2x^{-1}} + \underline{c_1x} - \underline{c_2x^{-1}} - \underline{c_1x} - \underline{c_2x^{-1}} \stackrel{?}{=} 0$$

collect like terms

$$x^{-1}(2c_2 - c_2 - c_2) + x(c_1 - c_1) \stackrel{?}{=} 0$$

$$x^{-1}(0) + x(0) \stackrel{?}{=} 0$$

$0 = 0$ ✓

$y = c_1x + \frac{c_2}{x}$ is twice diff'ble and the equation is true upon substitution.

Hence it is a solution.

Solutions: Implicit

Consider the ODE $F(x, y, y', \dots, y^{(n)}) = 0$ (2)

Definition: Implicit Solutions

Definition: An **implicit solution** of (2) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (2) and this relation.

Recall that a **relation** is an equation in the two variables x and y .
Something like

$$x^2 + y^2 = 4, \quad x \ln(y) = y^2 \cos(2x), \quad \text{or} \quad xy = e^y$$

would be examples of relations.

Example: Implicitly Defined Solution(s)

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \quad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll assume the relation holds and show it implies the ODE. Use implicit differentiation. Find $\frac{d}{dx}$ of the relation

$$\frac{d}{dx} (y^2 - 2x^2y) = \frac{d}{dx} (1)$$

$$2y \frac{dy}{dx} - 2 \left(2xy + x^2 \frac{dy}{dx} \right) = 0$$

It may not be possible to clearly identify the domain of definition of an implicit solution.

Let's solve for $\frac{dy}{dx}$. Cancel 2 and distribute minus sign.

$$y \frac{dy}{dx} - 2xy - x^2 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2xy}{y - x^2}$$

$$(y - x^2) \frac{dy}{dx} = 2xy$$

$$\text{for } y - x^2 \neq 0 \quad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

This is the expected ODE. So $y^2 - 2x^2y = 1$ defines a solution to

$$\frac{dy}{dx} = \frac{2xy}{y - x^2}$$

Function vs Solution

The interval of definition has to be an **interval**.

Consider the ODE

$$\frac{dy}{dx} = -y^2.$$

The function $y = \frac{1}{x}$ is a solution. The domain of $f(x) = \frac{1}{x}$

- ▶ as a **function** could be stated as $(-\infty, 0) \cup (0, \infty)$.
- ▶ as a **solution** to an ODE could be stated as $(0, \infty)$, or as $(-\infty, 0)$.

In the absence of additional information, we'll usually take the interval of definition to be the largest possible one (or one of the largest possible ones).

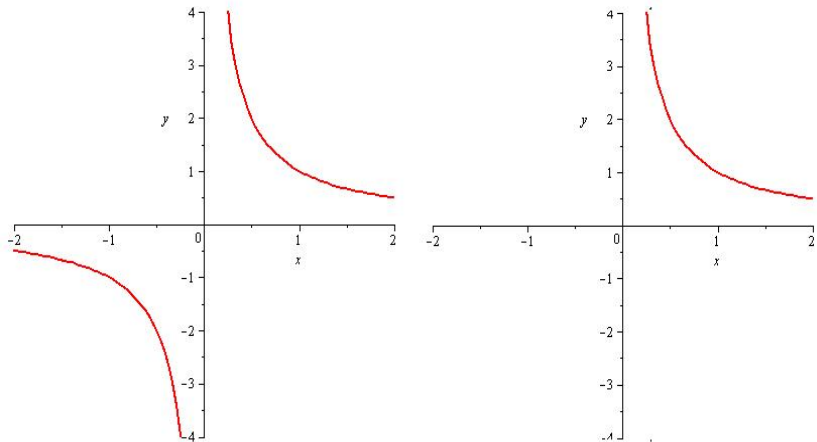


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE. The graph of the solution to an ODE will not have disjoint pieces.

Additional Common Terms

- ▶ A **parameter** is an unspecified constant such as c_1 and c_2 in $y = c_1x + \frac{c_2}{x}$.
- ▶ A **family of solutions** is a collection of solution functions that only differ by a parameter.
- ▶ An **n -parameter family of solutions** is one containing n parameters (e.g. $y = c_1x + \frac{c_2}{x}$ is a 2-parameter family).
- ▶ A **particular solution** is one with no arbitrary constants in it (i.e., no unspecified parameters). e.g. $y = 3x - \frac{17}{x}$
- ▶ The **trivial solution** is the simple constant function $y = 0$.
- ▶ An **integral curve** is the graph of one solution (perhaps from a family).

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- ▶ predator and prey
- ▶ competing species
- ▶ two or more masses attached to a system of springs
- ▶ two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**. A system is linear if each equation in the system is a linear equation.

Example of Nonlinear System

Lotka-Volterra Model

$$\begin{aligned}\frac{dx}{dt} &= -\alpha x + \beta xy \\ \frac{dy}{dt} &= \gamma y - \delta xy\end{aligned}$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α , β , γ and δ are nonnegative constants.

This model is built on the assumptions that

- ▶ in the absence of predation, prey increase exponentially
- ▶ in the absence of predation, predators decrease exponentially,
- ▶ predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

LR-Circuit Network Example

$$\frac{di_2}{dt} = -2i_2 - 2i_3 + 60$$

$$\frac{di_3}{dt} = -2i_2 - 5i_3 + 60$$

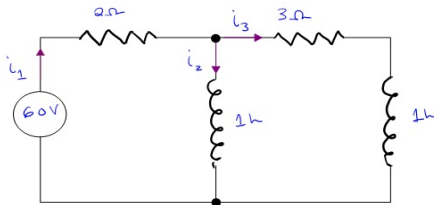


Figure: Electrical Network of resistors and inductors showing currents i_2 and i_3 modeled by this system of equations.

Solutions to Systems of ODE

A **solution** to a system of equations requires a function for each dependent variable. These functions must reduce every equation^a in the system to an identity upon substitution.

^aThe point being that all equations are considered together so that all must be satisfied.

Example: Show that the pair of functions $i_2(t) = 30 - 24e^{-t} - 6e^{-6t}$ and $i_3(t) = 12e^{-t} - 12e^{-6t}$ are a solution to the system

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60 \\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

Exercise left to the reader.

Systems of ODEs

Solution Methods

There are various approaches to solving a system of differential equations. These can include

- ▶ elimination (try to eliminate a dependent variable),
- ▶ matrix techniques,
- ▶ Laplace transforms .
- ▶ numerical approximation techniques

We will use Laplace transforms to solve select systems of linear equations later in the course.