August 16 Math 2306 sec. 51 Spring 2023

Section 1: Concepts and Terminology

Our first goal is to define what a *differential equation* is and identify relevant properties and the language used to talk about this subject.

A Differential Equation

Suppose $y = \phi(x)$ is a differentiable function. We know that $\frac{dy}{dx} = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$. Note that this function happens to satisfy the equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

$y'' = -4\cos(2x)$ $\quad$ $4y = 4\cos(2x)$

$\Rightarrow \quad y'' + 4y = -4\cos(2x) + 4\cos(2x) = 0$

A Differential Equation

The equation $\frac{d^2y}{dx^2} + 4y = 0$ is an example of a **differential equation**.

A couple of immediate questions arise:

Questions

1. If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it?
2. Also, is $\cos(2x)$ the only possible function that y could be?

We'll be able to answer these and many more questions as we proceed.

Definition

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

We can characterize the *variables* in calculus terms:

Independent and Dependent Variables

An **Independent Variable:** will appear as one that derivatives are taken with respect to. (The x in $y = f(x)$.)
A **Dependent**[a] **Variable:** will appear as one that derivatives are taken of. (The y in $y = f(x)$.)

[a] Think of dependent variables as the functions.

Independent and Dependent Variables

Identify the independent and dependent variables in each differential equation shown.

$\dfrac{d^2y}{dx^2} + 4y = 0$	Independent $\underline{\quad x \quad}$	Dependent $\underline{\quad y \quad}$
$t^2y'' - 2ty' + y = \sin(t)$	Independent $\underline{\quad t \quad}$	Dependent $\underline{\quad y \quad}$
$2\dfrac{d^2x}{du^2} = x^2 + u^2$	Independent $\underline{\quad u \quad}$	Dependent $\underline{\quad x \quad}$

Sometimes the independent variable isn't explicitly stated, e.g., consider $y'' + 4y = 0$. I'll usually just call it x or maybe t.

Classifications: Type ODE or PDE

ODEs

An **ordinary differential equation (ODE)** has exactly one independent variable[a]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

[a] These are the subject of this course.

PDEs

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Note on Notation

The subject of this course is **Ordinary Differential Equations.** So we won't be considering PDEs. The symbol ∂ is called a *partial* symbol. It is used to express derivatives when there are two or more independent variables. It's similar to d but indicates that one variable is being held fixed. If the function $u(x, t)$ depends on two variables x and t, then the expression

$$\frac{\partial u}{\partial t}$$

is read as "the "partial derivative of u with respect to t." It is defined by

$$\lim_{\Delta t \to 0} \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}.$$

Classifications: Order

Definition

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

Examples: Identify the order of each differential equation.

$$\frac{dy}{dx} - y^2 = 3x$$

Order = 1st ODE

$$y''' + (y')^4 = x^3$$

Order = 3rd ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

Order = 2nd PDE

Notations and Symbols

We'll mostly use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \dots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \dots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

On occasion, we'll want to reference a generic ODE. We have a couple of formats for that.

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n+2$ variables.

Example: Express the equation $y'' + 4y = 0$ in the form

$F(x, y, y', \ldots, y^{(n)}) = 0$

This is already in this form.

$$F(x, y, y', y'') = y'' + 4y.$$

Notations and Symbols

Normal Form

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

E.g. if $n = 1$ $\quad \frac{dy}{dx} = f(x, y)$, $\qquad$ if $n = 2$ $\quad \frac{d^2 y}{dx^2} = f(x, y, y')$.

Example: Express the equation $y'' + 4y = 0$ in the normal form.

In normal form the ODE is

$y'' = -4y$, $\qquad f(x, y, y') = -4y$

Notations and Symbols

Differential Form

If $M(x, y)$ and $N(x, y)$ are functions of the variables x and y, then an expression of the form

$$M(x, y)\,dx + N(x, y)\,dy$$

is called a **Differential Form.** A first order equation may be written in terms of a differential form as follows:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Note that this can be rearranged into a couple[1] of different normal forms

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \quad \text{or} \quad \frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}$$

[1] We have to assume that N or M is nonzero as needed.

Classifications: Linearity

Linear Differential Equation

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Example First Order:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Example Second Order:

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Properties of a Linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Properties of Linear ODEs

- Each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not the dependent one.
- y, and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- The characteristic structure of the left side is that

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \quad \ldots, \quad \frac{d^n y}{dx^n}$$

  are multiplied by functions of the independent variable and added together.

Examples (Linear -vs- Nonlinear)

Verify that the two equations here are **linear.**

1. $$y'' + 4y = 0 \qquad\qquad t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$a_0(x) = 4$ $\qquad$ $a_0(t) = -1$

$a_1(x) = 0$ $\qquad$ $a_1(t) = 2t$

$a_2(x) = 1$ $\qquad$ $a_2(t) = t^2$

$g(x) = 0$ $\qquad$ $g(t) = e^t$

Examples (Linear -vs- Nonlinear)

Determine why the following two equations are **nonlinear.**

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$u'' + u' = \cos u$$

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^3 \frac{dy}{dx} = x^3$$

u is dependent

$(y')^4$ is a nonlinear term

$\cos u$ is a nonlinear term

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

rearrange

$$y''' + y'' + 2ty' - y = \cos t$$

- independent var. t
- dependent var. y
- order 3rd
- linear/nonlinear linear

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(b) $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ g and ℓ are constant

Sin θ is a nonlinear term.

for $|\theta| << 1$

$\sin\theta \approx \theta$

- independent var. time
- dependent var. θ
- order 2^{nd}
- linear/nonlinear nonlinear