

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2 \sin(2x).$$

Even dy/dx is differentiable with $\frac{d^2y}{dx^2} = -4 \cos(2x)$.

Suppose $y = \cos(2x)$

Note that $\frac{d^2y}{dx^2} + 4y = 0$.

$$y'' = -4 \cos(2x)$$

$$y'' + 4y = -4 \cos(2x) + 4 \cos(2x) = 0$$

A differential equation

The equation $\frac{d^2y}{dx^2} + 4y = 0$

is an example of a **differential equation**.

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it?

Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more independent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

In $y = f(x)$, the y is dependent and the x is independent.

Classifications (ODE versus PDE)

Type: An **ordinary differential equation (ODE)** has exactly one independent variable¹. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t,$$

$$\text{or} \quad y'' + 4y = 0$$

¹These are the subject of this course.

Classifications (ODE versus PDE)

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

The expression $\frac{\partial y}{\partial t}$ is read

the partial derivative of y with respect to t .

It's computed by taking the derivative of $y = f(x, t)$ while keeping the other variable, x , fixed.

Classifications (Order)

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

1st order ODE

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

Leibniz: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n},$ or

Prime & superscripts: $y', y'', \dots y^{(n)}.$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

velocity is $\frac{ds}{dt} = \dot{s},$ and acceleration is $\frac{d^2s}{dt^2} = \ddot{s}$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Our equation $\frac{d^2y}{dx^2} + 4y = 0$ has this form where

$$F(x, y, y', y'') = \frac{d^2y}{dx^2} + 4y.$$

Notations and Symbols

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

Our equation $\frac{d^2 y}{dx^2} + 4y = 0$ can be written in normal form.

$$\frac{d^2 y}{dx^2} = -4y \quad \text{note that} \quad f(x, y, y') = -4y$$

Classifications (Linear Equations)

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

Example First Order:

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Example Second Order:

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Properties of a Linear ODE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

- ▶ Each of the coefficients a_0, \dots, a_n and the right hand side g may depend on the independent variable but not the dependent one.
- ▶ y , and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- ▶ The characteristic structure of the left side is that

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \dots, \frac{d^n y}{dx^n}$$

are multiplied by functions of the independent variable and added together.

Examples of Linear ODEs

$$y'' + 4y = 0$$

$$t^2 \frac{d^2 x}{dt^2} + 2t \frac{dx}{dt} - x = e^t$$

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_0(x) = 4$$

$$a_1(x) = 0$$

$$a_2(x) = 1$$

$$g(x) = 0$$

$$a_0(t) = -1$$

$$a_1(t) = 2t$$

$$a_2(t) = t^2$$

$$g(t) = e^t$$

Examples of Nonlinear ODEs

$$\frac{d^3 y}{dx^3} + \left(\frac{dy}{dx} \right)^4 = x^3 .$$

$$y''' + (y')^3 y' = x^3$$

$(y')^4$ is a
nonlinear
term

$y''' = x^3$
would
be linear
 $y''' + y' = x^3$

$$u'' + u' = \cos u$$

↑
this is
nonlinear
because
 u is
dependent.

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$ *rearrange*

$$y''' + y'' + 2ty' - y = \cos t$$

Dependent = y

Independent = t

Order = 3rd

Linear? Yes it's linear

(b) $\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0$ g and ℓ are constant

Dep θ = θ

Indep. = time (t)

Order = 2nd

Linear ?

Nonlinear due
to $\sin \theta$

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Solution or Explicit Solution

Definition: A function ϕ defined on an interval² I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Example: $\phi(x) = \cos(2x)$ is a **solution** of $y'' + 4y = 0$ on $(-\infty, \infty)$ because

- ▶ it is twice differentiable, and
- ▶ when we set $y = \cos(2x)$ in the equation, it gives a true statement (namely, $0 = 0$).

²The interval is called the *domain of the solution* or the *interval of definition*. 

Examples:

Verify that the given function is an solution of the ODE on the indicated interval. The c_1 and c_2 are constants.

$$\phi(x) = c_1 x + \frac{c_2}{x}, \quad I = (0, \infty), \quad x^2 y'' + xy' - y = 0$$

① ϕ has 2 continuous derivatives on I

② Sub $y = \phi$ into the ODE.

$$y = c_1 x + \frac{c_2}{x}$$

$$y' = c_1 - \frac{c_2}{x^2}$$

$$y'' = 0 + \frac{2c_2}{x^3}$$

$$x^2 y'' + x y' - y \stackrel{?}{=} 0$$

$$x^2 \left(\frac{2c_2}{x^3} \right) + x \left(c_1 - \frac{c_2}{x^2} \right) - \left(c_1 x + \frac{c_2}{x} \right) \stackrel{?}{=} 0$$

$$\frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} \stackrel{?}{=} 0$$

$$x(c_1 - c_1) + \frac{1}{x}(2c_2 - c_2 - c_2) \stackrel{?}{=} 0$$

$$x(0) + \frac{1}{x}(0) \stackrel{?}{=} 0$$

$$0 = 0$$

Hence $\phi = c_1 x + \frac{c_2}{x}$ is a solution.