August 18 Math 2306 sec. 51 Fall 2021

Section 1: Concepts and Terminology

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

When Leibniz notation is used, it's obvious what the variable names are. The expression $\frac{dy}{dx}$ implies that y is some function of x making y a dependent variable and x an independent one.

Sometimes it's not obvious what the independent variable is. In which case, we'll give it a name and be satisfied with that.

$$y'' + 4y = 0$$

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

∂ partial symbol

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

1st derivative

$$\frac{dy}{dx} - y^2 = 3x$$

1st order ODE

3rd 1st

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:}\quad \frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \dots\quad \frac{d^ny}{dx^n},\quad \text{or}$$

$$\text{Prime \& superscripts:}\quad y',\quad y'',\quad \dots\quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n+2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

This is in the form

$$F(x, y, y', y'') = 0$$

Here $F(x, y, y', y'') = y'' + 4y$

In normal form, the ODE is

$$\frac{d^2y}{dx^2} = -4y$$

The

$f(x, y, y') = -4y$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$\underbrace{M(x, y)\,dx + N(x, y)\,dy}_{\text{Differential form}} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

$$N(x,y)\,dy = -M(x,y)\,dx \qquad dy = \frac{dy}{dx}\,dx$$

Divide by N and dx

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} \qquad (\text{for } N \neq 0)$$

or $$\frac{dx}{dy} = -\frac{N(x,y)}{M(x,y)}$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

- Each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not the dependent one.
- y, and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- The characteristic structure of the left side is that
$$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$$
are multiplied by functions of the independent variable and added together.

Examples (Linear -vs- Nonlinear)

Both linear

$$y'' + 4y = 0$$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$a_2(x) = 1$

$a_1(x) = 0$

$a_0(x) = 4$

$g(x) = 0$

$$t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$

$a_2(t) = t^2$

$a_1(t) = 2t$

$a_0(t) = -1$

$g(t) = e^t$

Examples (Linear -vs- Nonlinear)

Both nonlinear

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$y''' + (y')^3 y' = x^3$$

nonlinear term

$$u'' + u' = \cos u$$

nonlinear term

u is dependent

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

Independent $\underline{\ t\ }$

dependent $\underline{\ y\ }$

order 3rd

$y''' + y'' + 2ty' - y = \cos t$

linear

(b) $\ddot{\theta} + \frac{g}{\ell} \sin \theta = 0$ g and ℓ are constant

Independent time e.s. t

dependent θ

order 2^{nd}

nonlinear due to $\sin \theta$ term.

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[2] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[2] The interval is called the *domain of the solution* or the *interval of definition*.