August 18 Math 2306 sec. 52 Spring 2023

Section 1: Concepts and Terminology

Thus far, we've

- defined *differential equation* and identified dependent and independent variables,
- classified ordinary (ODE) and partial (PDE) differential equations,
- classified equations by **order**,
- and characterized *linear* versus *nonlinear* ODEs.

Now, we want to define **solutions**. Solutions come in two basic flavors, but the essence of solving a differential equation is identifying the function(s) that make the equation true. Let's define solutions of two types.

Solution

$$\text{Consider the ODE} \quad F(x, y, y', \ldots, y^{(n)}) = 0 \tag{1}$$

Definition: Solution (*Explicit* Solution)

A function ϕ defined on an interval I[a] and possessing at least n continuous derivatives on I is a **solution** of (1) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

[a] The interval is called the *domain of the solution* or the *interval of definition*.

Solution means what you'd expect. A solution is a function that satisfies, makes the differential equation, true.

Solution Example

Verify that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation on the interval $(0, \infty)$.

$$x^2 y'' + xy' - y = 0$$

$y = c_1 x + \frac{c_2}{x}$ this is twice continuously differentiable on $(0, \infty)$.

$y = c_1 x + c_2 x^{-1}$

$y' = c_1 - c_2 x^{-2}$

$y'' = 2 c_2 x^{-3}$

$$x^2 y'' + x y' - y \stackrel{?}{=} 0$$

$$x^2(2c_2x^{-3}) + x(c_1 - c_2x^{-2}) - (c_1x + c_2x^{-1}) \stackrel{?}{=} 0$$

$$2c_2x^{-1} + c_1x - c_2x^{-1} - c_1x - c_2x^{-1} \stackrel{?}{=} 0$$

$$x^{-1}(2c_2 - c_2 - c_2) + x(c_1 - c_1) \stackrel{?}{=} 0$$

$$x^{-1}(0) + x(0) \stackrel{?}{=} 0$$

$$0 = 0$$

an identity

This shows that $y = c_1x + \frac{c_2}{x}$ is a soln. of $x^2y'' + xy' - y = 0$ on $(0, \infty)$.

Solution: Implicit

Consider the ODE $F(x, y, y', \dots, y^{(n)}) = 0$ (2)

Definition: Implicit Solutions

Definition: An **implicit solution** of (2) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (2) and this relation.

Recall that a **relation** is an equation in the two variables x and y. Something like

$$x^2 + y^2 = 4, \quad x\ln(y) = y^2\cos(2x), \quad \text{or} \quad xy = e^y$$

would be examples of relations.

Example: Implicitly Defined Solution(s)

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll assume the relation is true and show that it implies the ODE. We use implicit differentiation

$y^2 - 2x^2y = 1$, take $\frac{d}{dx}$ of both sides assuming y is a fnct. of x.

$$\frac{d}{dx}\left(y^2 - 2x^2y\right) = \frac{d}{dx}(1) = 0$$

It may not be possible to clearly identify the domain of definition of an implicit solution.

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

Isolate $\frac{dy}{dx}$:

$$\frac{1}{2}\left(2y\frac{dy}{dx} - 2\left(2xy + x\frac{dy}{dx}\right)\right) = \frac{1}{2}(0)$$

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy$$

$$(y - x^2)\frac{dy}{dx} = 2xy$$

for $y - x^2 \neq 0$ $\quad \frac{dy}{dx} = \frac{2xy}{y - x^2}$

This shows that $y^2 - 2x^2y = 1$ defines a solution for $\frac{dy}{dx} = \frac{2xy}{y - x^2}$

Function vs Solution

The interval of definition has to be an interval.

Consider the ODE

$$\frac{dy}{dx} = -y^2.$$

The function $y = \dfrac{1}{x}$ is a solution. The domain of $f(x) = \dfrac{1}{x}$

- as a **function** could be stated as $(-\infty, 0) \cup (0, \infty)$.
- as a **solution** to an ODE could be stated as $(0, \infty)$, or as $(-\infty, 0)$.

In the absence of additional information, we'll usually take the interval of definition to be the largest possible one (or one of the largest possible ones).

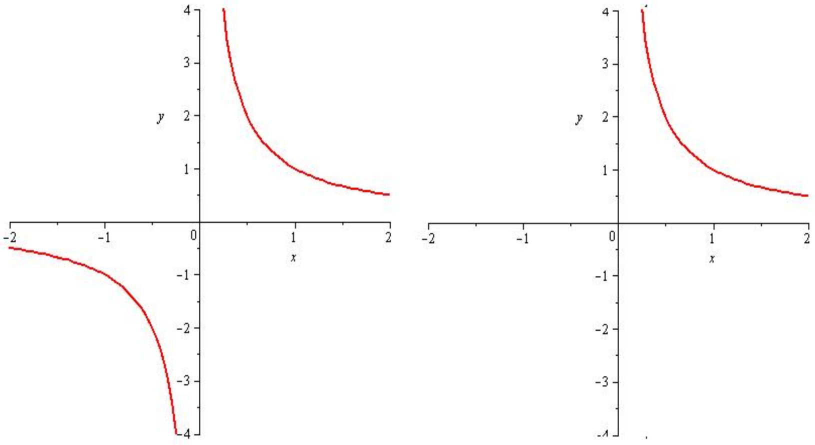

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE. The graph of the solution to an ODE will not have disjoint pieces.

Additional Common Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in $y = c_1 x + \frac{c_2}{x}$.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family).
- A **particular solution** is one with no arbitrary constants in it (i.e., no unspecified parameters).
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- predator and prey
- competing species
- two or more masses attached to a system of springs
- two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**. A system is linear if each equation in the system is a linear equation.

Example of Nonlinear System

Lotka-Volterra Model

$$\frac{dx}{dt} = -\alpha x + \beta xy$$
$$\frac{dy}{dt} = \gamma y - \delta xy$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α, β, γ and δ are nonnegative constants.
This model is built on the assumptions that

- in the absence of predation, prey increase exponentially
- in the absence of predation, predators decrease exponentially,
- predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

LR-Circuit Network Example

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60\\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

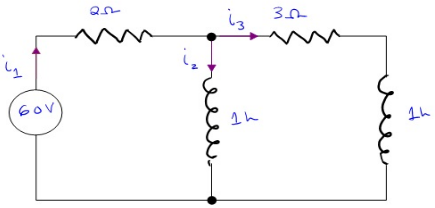

Figure: Electrical Network of resistors and inductors showing currents i_2 and i_3 modeled by this system of equations.

Solutions to Systems of ODE

A **solution** to a system of equations requires a function for each dependent variable. These functions must reduce every equation[a] in the system to an identity upon substitution.

[a]The point being that all equations are considered together so that all must be satisfied.

Example: Show that the pair of functions $i_2(t) = 30 - 24e^{-t} - 6e^{-6t}$ and $i_3(t) = 12e^{-t} - 12e^{-6t}$ are a solution to the system

$$\begin{aligned} \frac{di_2}{dt} &= -2i_2 - 2i_3 + 60 \\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60 \end{aligned}$$

Exercise left to the reader.

Systems of ODEs

Solution Methods

There are various approaches to solving a system of differential equations. These can include

- elimination (try to eliminate a dependent variable),
- matrix techniques,
- Laplace transforms
- numerical approximation techniques

We will use Laplace transforms to solve select systems of linear equations later in the course.