August 18 Math 2306 sec. 54 Fall 2021

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$.

Suppose $y = \cos(2x)$ ← this function makes this equation true

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

for $y = \cos(2x)$ so $y'' = -4\cos(2x)$

substituting

$$y'' + 4y = -4\cos(2x) + 4\cos(2x)$$
$$= 0$$

A *differential* equation

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

these are functions

Independent and Dependent Variables

Often, the derivatives indicate which variable is which:

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

dependent

independent

implies y is a function of x

dependent

independent

x is a function of r.

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

∂ partial symbol

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$ $\leftarrow$ 1st derivative

1st order ODE

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n+2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

$$\frac{d^2y}{dx^2} + 4y = 0$$

This has the form $F(x, y, y', y'') = 0$

Here $F(x, y, y', y'') = y'' + 4y$

The ODE in normal form ~~is~~

$$\frac{d^2y}{dx^2} = -4y$$

Here $f(x, y, y') = -4y$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$\underbrace{M(x, y)\, dx + N(x, y)\, dy = 0}_{\text{differential form}}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Differential forms may be written in normal form in a couple of ways.

$N(x,y)\,dy = -M(x,y)\,dx$ $\qquad\qquad dy = \frac{dy}{dx}\,dx$

Divide by $N(x,y)\,dx$ to get

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} \quad \text{for} \quad N(x,y) \neq 0$$

or

$$\frac{dx}{dy} = \frac{-N(x,y)}{M(x,y)} \quad \text{for} \quad M(x,y) \neq 0$$

Classifications

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

- Each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not the dependent one.
- y, and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- The characteristic structure of the left side is that

  $$y, \quad \frac{dy}{dx}, \quad \frac{d^2 y}{dx^2}, \ldots, \frac{d^n y}{dx^n}$$

  are multiplied by functions of the independent variable and added together.

Examples (Linear -vs- Nonlinear)

Both linear

$$y'' + 4y = 0 \qquad\qquad t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$a_2(x) = 1$

$a_1(x) = 0$

$a_0(x) = 4$

$g(x) = 0$

$a_2(t)x'' + a_1(t)x' + a_0(t)x = g(t)$

$a_2(t) = t^2$

$a_1(t) = 2t$

$a_0(t) = -1$

$g(t) = e^t$

Examples (Linear -vs- Nonlinear)

Both nonlinear

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

non linear term

$$u'' + u' = \cos u$$

nonlinear term

u is dependent