

August 18 Math 3260 sec. 51 Fall 2025

Chapter 1: The Vector Spaces R^n

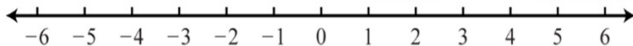
Our first goal is to understand the spaces of real n -tuples in the sense of an algebraic structure. We will use the notation

$$R^n \quad (\text{read "ar - en"})$$

to denote the set of ordered n -tuples of real numbers. Starting with $n = 1$

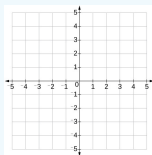
R^1 (or simply R)

The set of real 1-tuples—a.k.a. the set of real numbers. We often depict R geometrically as a line, where we place a marker for zero (the origin). Positive numbers are positioned to the right of the origin with negative numbers to the left.



R^2

The set of real, ordered pairs (x_1, x_2) , typically depicted geometrically as a plane with two axes that meet at $(0, 0)$, the origin.



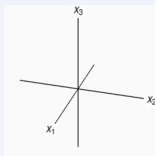
An element, a.k.a. a *point*, in R^2 has two coordinates^a, and order matters. For example, $(1, 3)$ is not the same as $(3, 1)$.

These are often written (x, y) , but we will mostly stick with the convention of using one character with subscripts, (x_1, x_2) .

^aWe'll use terms like *entry*, *component*, or *coordinate* to refer to each number in our n -tuple.

\mathbb{R}^3

The set of real, ordered triples (x_1, x_2, x_3) . These can also be visualized geometrically by considering a set of three coordinate axes.



Here too, the element is ordered so that $(1, -1, 4)$ is distinct from $(-1, 4, 1)$, and so forth.

These are often written (x, y, z) in the Calculus setting, but here we'll use (x_1, x_2, x_3) or perhaps (u_1, u_2, u_3) .

R^n

For $n \geq 4$, we lose the nice visual representations. But we can consider ordered 4-tuples, 5-tuples, and so forth.

$$\underbrace{(x_1, x_2, x_3, x_4)}_{R^4} \quad \underbrace{(x_1, x_2, x_3, x_4, x_5)}_{R^5} \quad \cdots \quad \underbrace{(x_1, x_2, \dots, x_n)}_{R^n}$$

We want to define algebraic operations on elements of R^n , consider properties of those operations, and consider some geometric implications.

We'll start by developing everything in R^2 —to take advantage of pictures. Then we'll extend these ideas in a natural way to R^3 , and more generally to R^n .

Our first task is to define a **vector** in R^2 .

Directed Line Segements

Consider a pair of points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$. The object

$$\overrightarrow{PQ}$$

is the directed line segment starting at P and terminating at Q .

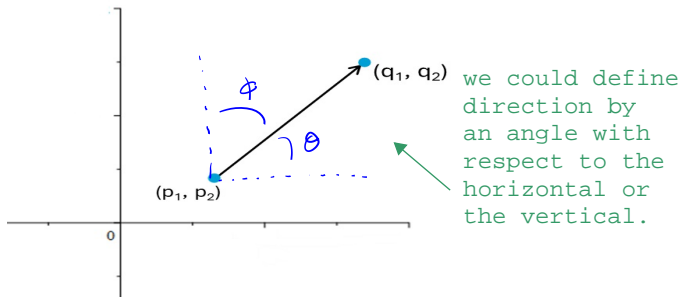


Figure: A directed line segment has two characteristic feature, a length and a direction.

Directed Line Segments

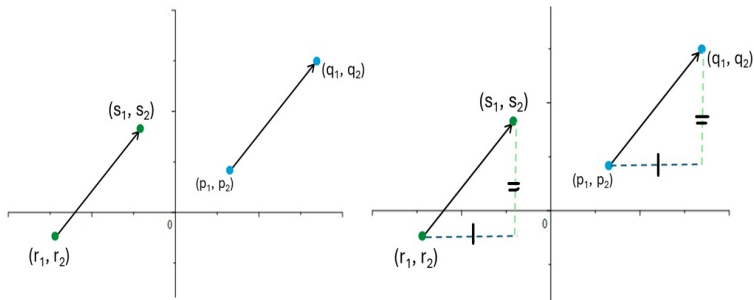


Figure: We will say that two directed line segments are equal provided they have the same length and direction.

Definition: Vector in R^2

A **vector** in R^2 is an ordered pair of real numbers,

$$\vec{x} = \langle x_1, x_2 \rangle,$$

that describe a length, called a *magnitude*, and a direction. The real numbers, x_1 and x_2 , are called the **entries** or **components** of the vector.

Remark 1: We will distinguish between *points* and *vectors* by using different delimiters.

a point (x_1, x_2) a vector $\langle x_1, x_2 \rangle$

And we'll distinguish between variables that represent a real number and a vector by placing a small arrow over a vector.

a number x a vector \vec{x}

Remark: Both points and vectors are 2-tuples. A point has a fixed position, relative to some coordinate system, whereas a vector does not.

Standard Representation

$\vec{x} = \langle x_1, x_2 \rangle = \overrightarrow{OX}$, where $O = (0, 0)$ and $X = (x_1, x_2)$.

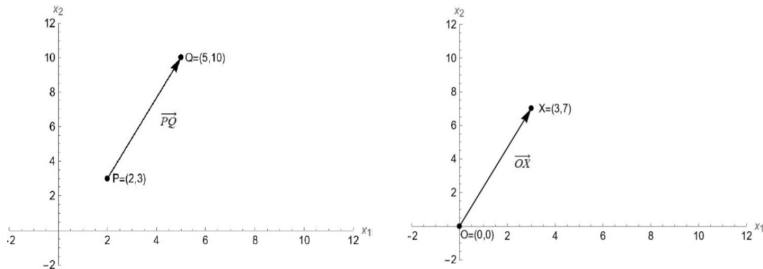


Figure: The vector $\langle 3, 7 \rangle$ can be represented by the directed line segment between $P = (2, 3)$ and $Q = (5, 10)$ (left). It's **standard representation** is the directed line segment between $O = (0, 0)$ and $X = (3, 7)$.

Remark: If $\vec{x} = \langle x_1, x_2 \rangle = \overrightarrow{PQ}$, $P = (p_1, p_2)$, and $Q = (q_1, q_2)$, then

$$x_i = q_i - p_i, \quad i = 1, 2.$$

Example:

$$\text{If } \vec{x} = \langle -3, 6 \rangle = \overrightarrow{PQ},$$

1. find Q if $P = (4, 15)$

$$x_1 = q_1 - p_1 \Rightarrow -3 = q_1 - 4 \Rightarrow q_1 = -3 + 4 = 1$$

$$x_2 = q_2 - p_2 \Rightarrow 6 = q_2 - 15 \Rightarrow q_2 = 6 + 15 = 21$$

$$Q = (1, 21)$$

2. find P if $Q = (7, -8)$

$$x_1 = q_1 - p_1 \quad -3 = 7 - p_1 \Rightarrow p_1 = 7 - (-3) = 10$$

$$x_2 = q_2 - p_2 \quad 6 = -8 - p_2 \Rightarrow p_2 = -8 - 6 = -14$$

$$P = (10, -14)$$

$$\langle x_1, x_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$$

Scalars

We will define several operations that we will use to manipulate vectors algebraically. Along with **vectors** we will work with **scalars**.

- ▶ A **vector** quantity has magnitude and direction.
- ▶ A **scalar** quantity has only magnitude

The set of scalars that we will use in this class is the set of real numbers, R .

In a more general setting, a set of **scalars** is something called a **field**. This means that they are subject to the operations of addition, subtraction, multiplication, and division following rules that we would recognize.