August 18 Math 3260 sec. 53 Fall 2025

Chapter 1: The Vector Spaces R^n

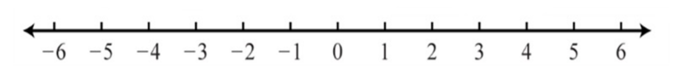
Our first goal is to understand the spaces of real n-tuples in the sense of an algebraic structure. We will use the notation

$$R^n \qquad \text{(read "ar - en")}$$

to denote the set of ordered n-tuples of real numbers. Starting with $n = 1$

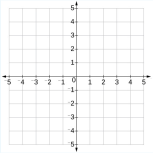
R^1 (or simply R)

The set of real 1-tuples—a.k.a. the set of real numbers. We often depict R geometrically as a line, where we place a marker for zero (the origin). Positive numbers are positioned to the right of the origin with negative numbers to the left.



R^2

The set of real, ordered pairs (x_1, x_2), typically depicted geometrically as a plane with two axes that meet at $(0, 0)$, the origin.



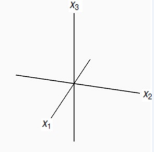
An element, a.k.a. a *point*, in R^2 has two coordinates[a], and order matters. For example, $(1, 3)$ is not the same as $(3, 1)$.

These are often written (x, y), but we will mostly stick with the convention of using one character with subscripts, (x_1, x_2).

[a] We'll use terms like *entry*, *component*, or *coordinate* to refer to each number in our n-tuple.

R^3

The set of real, ordered triples (x_1, x_2, x_3). These can also be visualized geometrically by considering a set of three coordinate axes.



Here too, the element is ordered so that $(1, -1, 4)$ is distinct from $(-1, 4, 1)$, and so forth.

These are often written (x, y, z) in the Calculus setting, but here we'll use (x_1, x_2, x_3) or perhaps (u_1, u_2, u_3).

R^n

For $n \geq 4$, we lose the nice visual representations. But we can consider ordered 4-tuples, 5-tuples, and so forth.

$$\underbrace{(x_1, x_2, x_3, x_4)}_{R^4} \qquad \underbrace{(x_1, x_2, x_3, x_4, x_5)}_{R^5} \quad \cdots \quad \underbrace{(x_1, x_2, \ldots, x_n)}_{R^n}$$

We want to define algebraic operations on elements of R^n, consider properties of those operations, and consider some geometric implications.

We'll start by developing everything in R^2—to take advantage of pictures. Then we'll extend these ideas in a natural way to R^3, and more generally to R^n.

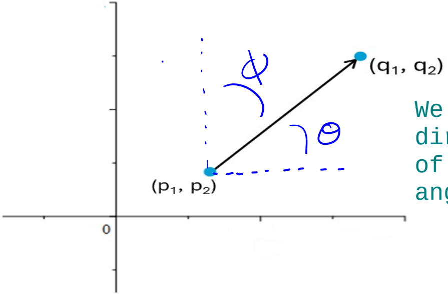
Our first task is to define a **vector** in R^2.

Directed Line Segements

Consider a pair of points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$. The object

$$\overrightarrow{PQ}$$

is the directed line segment starting at P and terminating at Q.



We can define direction in terms of one of these angles.

Figure: A directed line segment has two characteristic feature, a length and a direction.

Directed Line Segements

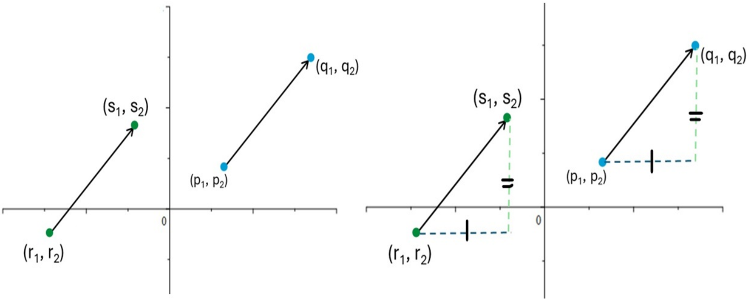


Figure: We will say that two directed line segments are equal provided they have the same length and direction.

Definition: Vector in R^2

A **vector** in R^2 is an ordered pair of real numbers,

$$\vec{x} = \langle x_1, x_2 \rangle,$$

that describe a length, called a *magnitude*, and a direction. The real numbers, x_1 and x_2, are called the **entries** or **components** of the vector.

Remark 1: We will distinguish between *points* and *vectors* by using different delimiters.

$$\text{a point} \quad (x_1, x_2) \qquad \text{a vector} \quad \langle x_1, x_2 \rangle$$

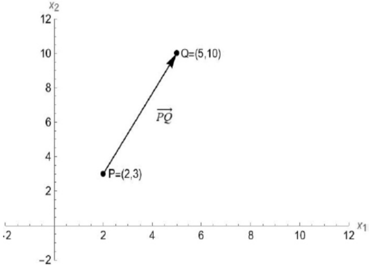
And we'll distinguish between variables that represent a real number and a vector by placing a small arrow over a vector.

$$\text{a number} \quad x \qquad \text{a vector} \quad \vec{x}$$

Remark: Both points and vectors are 2-tuples. A point has a fixed position, relative to some coordinate system, whereas a vector does not.

Standard Representation

$\vec{x} = \langle x_1, x_2 \rangle = \overrightarrow{OX}$, where $O = (0,0)$ and $X = (x_1, x_2)$.



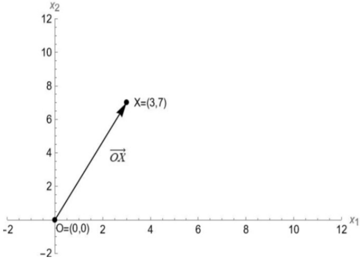


Figure: The vector $\langle 3, 7 \rangle$ can be represented by the directed line segment between $P = (2,3)$ and $Q = (5,10)$ (left). It's **standard representation** is the directed line segment between $O = (0,0)$ and $X = (3,7)$.

Remark: If $\vec{x} = \langle x_1, x_2 \rangle = \overrightarrow{PQ}$, $P = (p_1, p_2)$, and $Q = (q_1, q_2)$, then

$$x_i = q_i - p_i, \quad i = 1, 2.$$

Example:

If $\vec{x} = \langle -3, 6 \rangle = \overrightarrow{PQ}$,

1. find Q if $P = (4, 15)$

$x_1 = q_1 - p_1 \Rightarrow -3 = q_1 - 4 \Rightarrow q_1 = -3+4 = 1$

$x_2 = q_2 - p_2 \Rightarrow 6 = q_2 - 15 \Rightarrow q_2 = 6 + 15 = 21$

$Q = (1, 21)$

2. find P if $Q = (7, -8)$

$x_1 = q_1 - p_1 \Rightarrow -3 = 7 - p_1 \Rightarrow p_1 = 7 - (-3) = 10$

$x_2 = q_2 - p_2 \Rightarrow 6 = -8 - p_2 \Rightarrow p_2 = -8 - 6 = -14$

$P = (10, -14)$

$\langle x_1, x_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle$

Scalars

We will define several operations that we will use to manipulate vectors algebraically. Along with **vectors** we will work with **scalars.**

- A **vector** quantity has magnitude and direction.
- A **scalar** quantity has only magnitude

The set of scalars that we will use in this class is the set of real numbers, R.

In a more general setting, a set of **scalars** is something called a **field**. This means that they are subject to the operations of addition, subtraction, multiplication, and division following rules that we would recognize.

Algebra with Vectors

A **vectors space** always involves two primary operations:

- vector addition, and
- scalar multiplication.

First, let's define what is meant by **vector addition.**