August 20 Math 2306 sec. 52 Fall 2021

We have

- Defined what a *differential equation* is,
- distinguished **ordinary (ODE)** from **partial (PDE)** differential equations,
- defined the **order** of an ODE, and
- defined the classification of **linear** versus nonlinear.

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y''+2ty' = \cos t+y-y'''$

Indep var $\underline{\ t\ }$

depend var $\underline{\ y\ }$

order is $\underline{\ 3\ }$

$a(t)y''' + b(t)y'' + c(t)y' + d(t)y = f(t)$

$y''' + y'' + 2ty' - y = \cos t$

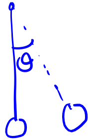
it is linear

(b) $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$ g and ℓ are constant

Indep. variable time (we could call it t)

depend variable θ

order 2



It's nonlinear

$a(t)\theta'' + b(t)\theta' + c(t)\theta = g(t)$

$\sin\theta$ is a nonlinear term since θ is dependent.

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval[1] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[1] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval.

$$\phi(t) = 3e^{2t}, \quad I = (-\infty, \infty), \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

The ODE is 2^{nd} order. $\phi = 3e^{2t}$ has derivatives of all orders on $(-\infty, \infty)$:

we'll substitute ϕ into the ODE. Set

$$y = \phi(t) = 3e^{2t}$$

We need y' and y''.

$$y = 3e^{2t}$$
$$y' = 3e^{2t}(2) = 6e^{2t}$$

$$y'' = 12e^{2t}$$

so $\quad y'' - y' - 2y \overset{?}{=} 0$

$$12e^{2t} - 6e^{2t} - 2(3e^{2t}) \overset{?}{=} 0$$

$$12e^{2t} - 6e^{2t} - 6e^{2t} =$$

$$0 = 0$$

we get a true statement, so $y = 3e^{2t}$ is a solution to $y'' - y' - 2y = 0$.

Example: Implicitly Defined Solution(s)

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We'll assume that the relation holds, i.e. x and y satisfy $y^2 - 2x^2y = 1$, and show that then y also satisfies the ODE. We'll use implicit differentiation.

$$\frac{d}{dx}\left(y^2 - 2x^2y\right) = \frac{d}{dx}(1)$$

$$2y\frac{dy}{dx} - 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

It may not be possible to clearly identify the domain of definition of an implicit solution.

Let's isolate $\frac{dy}{dx}$.

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y - x^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{y - x^2}$$

assuming $y - x^2 \neq 0$

Hence if y satisfies $y^2 - 2x^2y = 1$, then it also satisfies the ODE.

Function vs Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible intervasl. It may depend on other information.

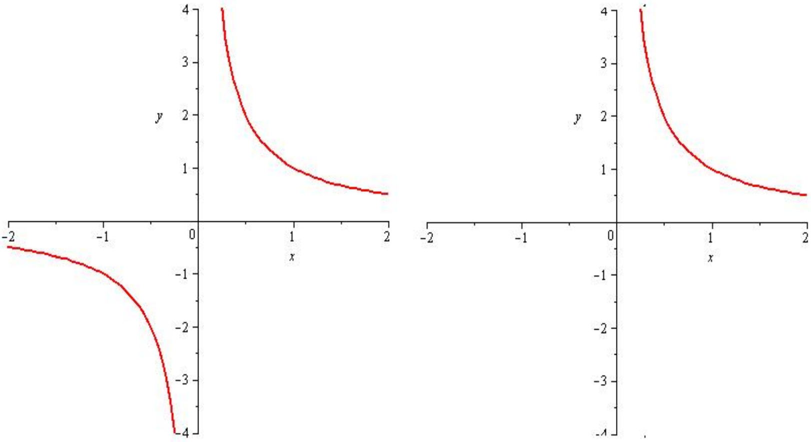


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- predator and prey
- competing species
- two or more masses attached to a system of springs
- two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**.

Example of Nonlinear System

$$\begin{aligned} \frac{dx}{dt} &= -\alpha x + \beta xy \\ \frac{dy}{dt} &= \gamma y - \delta xy \end{aligned}$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α, β, γ and δ are nonnegative constants.
This model is built on the assumptions that

- in the absence of predation, prey increase exponentially
- in the absence of predation, predators decrease exponentially,
- predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60\\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

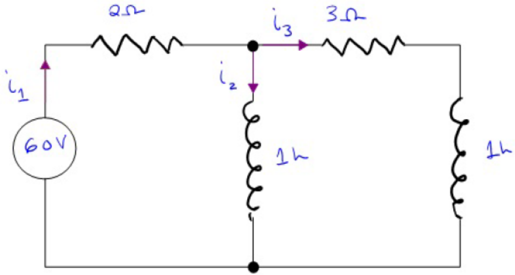


Figure: Electrical Network of resistors and inductors showing currents i_2 and i_3 modeled by this system of equations.

Systems of ODEs

There are various approaches to solving a system of differential equations. These can include

- elimination (try to eliminate a dependent variable),
- matrix techniques,
- Laplace transforms[2]
- numerical approximation techniques

[2] We'll consider this later.

Arbitrary Constants

Show that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

Substitute

$$y = c_1 x + c_2 x^{-1}$$

$$y' = c_1 - c_2 x^{-2}$$

$$y'' = 2c_2 x^{-3}$$

$$x^2 y'' + xy' - y \stackrel{?}{=} 0$$

$$x^2 (2c_2 x^{-3}) + x(c_1 - c_2 x^{-2}) - (c_1 x + c_2 x^{-1}) \stackrel{?}{=} 0$$

$$2c_2 x^{-1} + c_1 x - c_2 x^{-1} - c_1 x - c_2 x^{-1} \stackrel{?}{=} 0$$

Collect x and x^{-1}

$$x^{-1}(2c_2 - c_2 - c_2) + x(c_1 - c_1) =$$

$$0 = 0$$

So $y = c_1 x + \frac{c_2}{x}$ solves the ODE for any choice of c_1 and c_2.

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).