August 20 Math 3260 sec. 51 Fall 2025

Section 1.1 The Vector Space R^2

Section 1.1.1 What is a Vector in R^2

Definition: Vector in R^2

A **vector** in R^2 is an ordered pair of real numbers,

$$\vec{x} = \langle x_1, x_2 \rangle,$$

that describe a length, called a *magnitude*, and a direction. The real numbers, x_1 and x_2, are called the **entries** or **components** of the vector.

- Recall that we are using triangular brackets, $\langle \cdot, \cdot \rangle$, to distinguish a vector from a point, $(\cdot, \cdot)$.
- And we put an arrow hat on top of a variable representing a vector, $\vec{\cdot}$, to distinguish it from a non-vector variable.

Points & Vectors in R^2

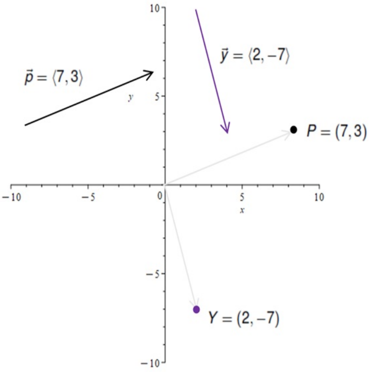


Figure: We can distinguish between points and vectors (which are both ordered pairs) by considering vectors as described by magnitude and direction as opposed to a fixed place relative to a coordinate system.

(The vector $\langle 0, 0 \rangle$ will be indistinguishable from a point.)

Algebra with Vectors

We'll work with two objects:

- **vectors** having magnitude and direction[a], and
- **scalars** having only *signed* magnitude.

The set of **scalars** used throughout this course will be the real numbers, R.

[a] There will be one special vector that doesn't have a defined direction.

A **vectors space** always involves two primary operations:

- vector addition, and
- scalar multiplication.

Section 1.1.2 Addition of Vectors

Given a pair of vectors, $\vec{x}$ and $\vec{y}$ in R^2, their sum, written

$$\vec{x} + \vec{y},$$

is a vector in R^2.

Vector Addition in R^2

Let $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$. Then the sum

$$\vec{x} + \vec{y} = \langle x_1 + y_1, x_2 + y_2 \rangle.$$

So vector addition is defined entry-wise.

Example

Find $\vec{u} + \vec{v}$ given

1. $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle 7, 4 \rangle$

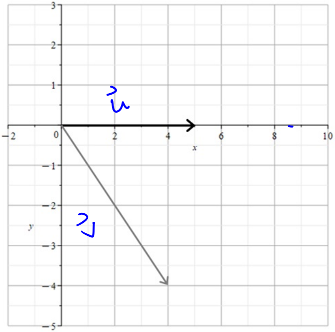
   $\vec{u} + \vec{v} = \langle 2+7, -3+4 \rangle = \langle 9, 1 \rangle$

2. $\vec{u} = \langle 5, 0 \rangle$ and $\vec{v} = \langle 4, -4 \rangle$

   $\vec{u} + \vec{v} = \langle 5+4, 0+(-4) \rangle = \langle 9, -4 \rangle$

Geometry of Vector Addition

Let's consider the sum $\vec{u} + \vec{v}$ where $\vec{u} = \langle 5, 0 \rangle$ and $\vec{v} = \langle 4, -4 \rangle$



$\vec{u} + \vec{v} = \langle 9, -4 \rangle$

Figure: Standard representations for $\vec{u} = \langle 5, 0 \rangle$ and $\vec{v} = \langle 4, -4 \rangle$.

We can slide the vector $\vec{v}$ so that its initial point is at the terminal point of $\vec{u}$.

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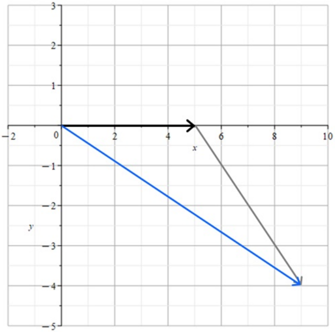


Figure: The standard representation of $\vec{u} + \vec{v}$ terminates at the new terminal point of $\vec{v}$.

Geometry of Vector Addition

Let's consider the sum $\vec{u} + \vec{v}$ where $\vec{u} = \langle 5, 0 \rangle$ and $\vec{v} = \langle 4, -4 \rangle$

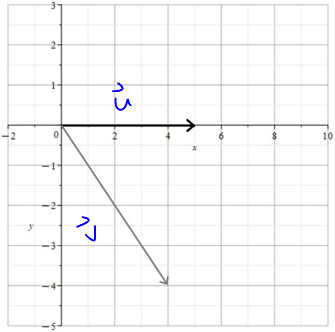


Figure: Standard representations for $\vec{u} = \langle 5, 0 \rangle$ and $\vec{v} = \langle 4, -4 \rangle$.

Instead, we can slide the vector $\vec{u}$ so that its initial point is at the terminal point of $\vec{v}$.

Geometry of Vector Addition

Let's consider the sum $\vec{u} + \vec{v}$ where $\vec{u} = \langle 5, 0 \rangle$ and $\vec{v} = \langle 4, -4 \rangle$

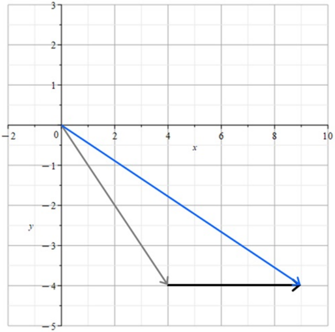


Figure: The standard representation of $\vec{u} + \vec{v}$ terminates at the new terminal point of $\vec{u}$.

Geometry of Vector Addition

Let's consider the sum $\vec{u}+\vec{v}$ where $\vec{u}=\langle 5,0\rangle$ and $\vec{v}=\langle 4,-4\rangle$

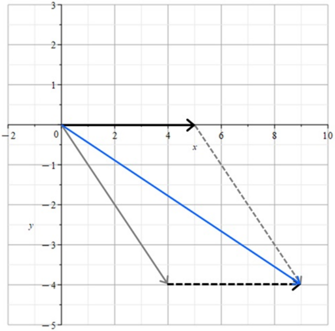


Figure: The standard representation of $\vec{u}+\vec{v}$ is the same in both cases.

Vector Addition is Commutative

For any pair $\vec{x}, \vec{y}$ in R^2,

$$\vec{x}+\vec{y}=\vec{y}+\vec{x}.$$

Three Vectors $\vec{x} = \langle 1, 1 \rangle, \quad \vec{y} = \langle 1, -2 \rangle, \quad \vec{z} = \langle 6, 3 \rangle$

$\vec{x} + \vec{y} + \vec{z} = \langle 8, 2 \rangle$

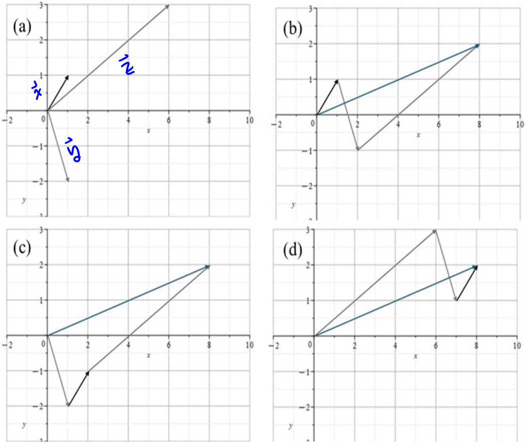


Figure: (a) standard representation of $\vec{x}$, $\vec{y}$, and $\vec{z}$, (b) $\vec{x} + \vec{y} + \vec{z}$, (c) $\vec{y} + \vec{x} + \vec{z}$, (d) $\vec{z} + \vec{y} + \vec{x}$.

Section 1.1.3 The Zero Vector & Additive Inverses

The Zero Vector

The vector in R^2 having both components zero is called the **zero vector** and is denoted

$$\vec{0}_2 = \langle 0, 0 \rangle.$$

The zero vector is the additive identity in the sense that

$$\vec{x} + \vec{0}_2 = \vec{0}_2 + \vec{x} = \vec{x}$$

for every vector $\vec{x}$ in R^2. The zero vector has zero length and is the only vector without a direction.

Additive Inverse

If $\vec{x} = \langle x_1, x_2 \rangle$ is any vector in R^2, then the **additive inverse** of $\vec{x}$ is the vector

$$-\vec{x} = \langle -x_1, -x_2 \rangle.$$

Note that $\vec{x} + (-\vec{x}) = -\vec{x} + \vec{x} = \vec{0}_2$. Vector subraction is simply addition of an additive inverse. So $\vec{x} - \vec{y} = \vec{x} + (-\vec{y})$.

Parallelogram Method of Vector Addition

If $\vec{x}$ and $\vec{y}$ are any nonzero and nonparallel vectors in R^2, then their standard representations define two sides of a parallelogram. One diagonal is the sum $\vec{x} + \vec{y}$. The differences, $\vec{x} - \vec{y}$ and $\vec{y} - \vec{x}$ are the other diagonal.

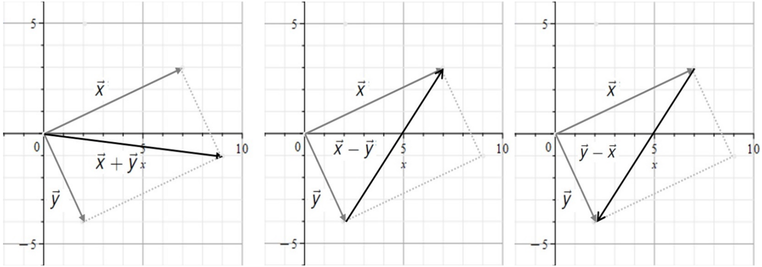


Figure: Parallelogram and diagonals defined by the vector sum and differences.

If the vectors are parallel, i.e., have the same or opposite (180°) directions, the parallelogram will be degenerate. It will be a line segment. (If both vectors are $\vec{0}_2$, then their sum and difference is a degenerate line (i.e., just a point).

Section 1.1.4 Scalar Multiples of Vectors

Given a vector $\vec{x}$ in R^2 and a scalar c in R, the scalar multiple $c\vec{x}$ is a vector in R^2.

Scalar Multiplication in R^2

Let $\vec{x} = \langle x_1, x_2 \rangle$ be a vector in R^2 and c a scalar (real number) Then the multiple

$$c\vec{x} = \langle cx_1, cx_2 \rangle.$$

So scalar multiplication is also defined entry-wise.

Example

Evaluate $c\vec{x}$ given

1. $\vec{x} = \langle 2, -3 \rangle$ and $c = 3$

$c\vec{x} = 3\langle 2, -3\rangle = \langle 3(2), 3(-3)\rangle = \langle 6, -9\rangle$

2. $\vec{x} = \langle 2, -3 \rangle$ and $c = -2$

$c\vec{x} = -2\langle 2, -3\rangle = \langle -2(2), -2(-3)\rangle = \langle -4, 6\rangle$

Geometry of Scalar Multiplication

Note that if $\vec{x} \neq \vec{0}_2$ and $c \neq 0$, the vector $c\vec{x}$

- points in the direction of $\vec{x}$ if $c > 0$ and
- points in the opposite direction (180° from) if $c < 0$.

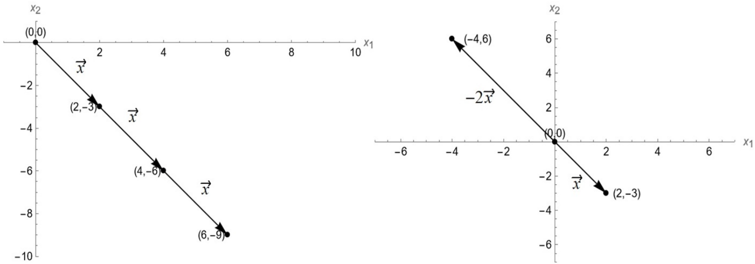


Figure: Example of scalar multiplication $c\vec{x}$ with $c > 0$ (left) and $c < 0$ (right).

Example

Let $\vec{x} = \langle 1, -3 \rangle$ and $\vec{y} = \langle 8, 12 \rangle$. Evaluate

1. $3\vec{x} - 2\vec{y}$

$3\vec{x} - 2\vec{y} = 3\langle 1, -3 \rangle - 2\langle 8, 12 \rangle$

$= \langle 3, -9 \rangle + \langle -16, -24 \rangle = \langle -13, -33 \rangle$

2. $4\vec{y} + 6\vec{x}$

$= 4\langle 8, 12 \rangle + 6\langle 1, -3 \rangle$

$= \langle 32, 48 \rangle + \langle 6, -18 \rangle = \langle 38, 30 \rangle$

Section 1.1.5 Linear Combinations in R^2

Linear Combination

Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a set of one or more ($k \geq 1$) vectors in R^2. A **linear combination** of these vectors is any vector of the form

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k,$$

where $c_1, \ldots, c_k$ are scalars. The coefficients, $c_1, \ldots, c_k$, are often called the **weights**. They can also be called **coefficients.**

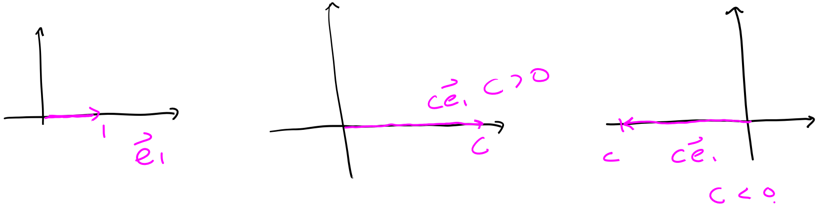
The big idea is that **linear combinations** is what we get when we use the two operations, vector addition and scalar multiplication. We are often interested in linear combinations in which we allow the weights to vary over some subset of R.

Example

Consider the vector $\vec{e}_1 = \langle 1, 0 \rangle$ in R^2. Give a geometric description of the set of all linear combinations of $\vec{e}_1$.

If $\vec{u}$ is a linear combination of $\vec{e}_1$ then

$\vec{u} = c\,\vec{e}_1 = c\langle 1, 0\rangle = \langle c, 0\rangle$ for some scalar c.



Letting c vary on $(-\infty, \infty)$, we can get every point on the

x_1- axis (i.e. the horizontal axis).

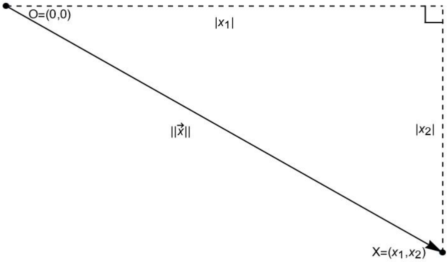
The terminal point of $c\vec{e}_1$ will lie on the horizontal axis. If c=0, we get the origin, if c > 0, we get the points on the right side of the origin, and if c < 0, we get the points to the left of the origin.

Section 1.1.6 Magnitude, Dot Product & Orthogonality

Magnitude

Let $\vec{x} = \langle x_1, x_2 \rangle$ be a vector in R^2. The magnitude of $\vec{x}$, denoted $\|\vec{x}\|$ is

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}.$$



Note $\|\vec{x}\|$ is a scalar

Figure: The magnitude of a vector is determined by the Pythagorean theorem.

Example

Find $\|\vec{x}\|$ for

1. $\vec{x} = \langle 7, -2 \rangle$ $\quad \|\vec{x}\| = \sqrt{7^2 + (-2)^2} = \sqrt{49+4} = \sqrt{53}$

2. $\vec{x} = \langle -3, 4 \rangle$ $\quad \|\vec{x}\| = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

3. $-2\vec{x}$ if $\vec{x} = \langle -3, 4 \rangle$

$-2\vec{x} = -2\langle -3, 4 \rangle = \langle 6, -8 \rangle$

$\|-2\vec{x}\| = \sqrt{6^2 + (-8)^2} = \sqrt{36+64} = \sqrt{100} = 10$

Scalar Multiplication & Magnitude

If $\vec{x}$ is in R^2 and c is any scalar, then

$$\|c\vec{x}\| = |c|\|\vec{x}\|.$$

Unit Vector

A vector $\vec{u}$ in R^2 is called a **unit vector** if

$$\|\vec{u}\| = 1.$$

Examples

Determine whether the given vector is a unit vector.

1. $\vec{x} = \langle 1, 1 \rangle$ $\quad \|\vec{x}\| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 1$

   $\vec{x}$ is not a unit vector

2. $\vec{y} = \langle \sin(27^\circ), \cos(27^\circ) \rangle$

   $\|\vec{y}\| = \sqrt{\sin^2(27^\circ) + \cos^2(27^\circ)} = 1$

   $\vec{y}$ is a unit vector

Examples

Find a scalar c such that $c\vec{x}$ is a unit vector if $\vec{x} = \langle 2, 2 \rangle$.

$\|c\vec{x}\| = |c| \, \|\vec{x}\|$ $\quad\quad$ $\|\vec{x}\| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$= |c|(2\sqrt{2}) = 1$

$\Rightarrow |c| = \dfrac{1}{2\sqrt{2}}$ $\quad\quad$ $c = \dfrac{1}{2\sqrt{2}}$ or $c = \dfrac{-1}{2\sqrt{2}}$

Parallel Vectors

We know that if $\vec{x} \neq \vec{0}_2$, and $c \neq 0$, then the vectors $\vec{x}$ and $c\vec{x}$ are either in the same direction or are in the opposite direction (e.g., 180°).

Parallel

We will say that two nonzero vectors $\vec{x}$ and $\vec{y}$ are parallel if there exists a scalar c such that

$$\vec{y} = c\vec{x}.$$

Example: If $\vec{x} = \langle 1, 2 \rangle$, determine whether the vector is parallel to $\vec{x}$ or not.

1. $\vec{y} = \langle -2, -4 \rangle$ $\langle -2, -4 \rangle \stackrel{?}{=} c \langle 1, 2 \rangle \Rightarrow$ yes if $c = -2$
2. $\vec{v} = \langle 2, 1 \rangle$ $\langle 2, 1 \rangle \stackrel{?}{=} c \langle 1, 2 \rangle \Rightarrow$ $c = 2$ and $c = \frac{1}{2}$ No, $\vec{v}$ is not ∥ to $\vec{x}$
3. $\vec{z} = \langle \frac{1}{2}, 1 \rangle$ Is $\vec{z} = c\vec{x}$? Yes $\vec{z} = \frac{1}{2}\vec{x}$