August 21 Math 2306 sec. 51 Fall 2024

Section 2: Initial Value Problems

We were considering **Euler's Method**, a method for obtaining an approximate solution to a first order IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Recall that the method amounts to building a sequence of approximations to the solution, $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, starting from the initial condition given and using tangent lines.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

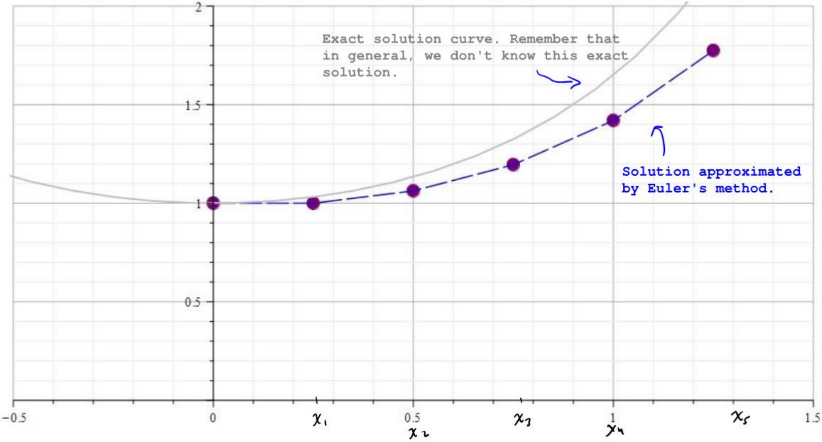

Figure: The dashed curve shows the approximation built using a step size of $h = 0.25$ to the true solution (in gray) to the IVP $y' = xy$ subject to $y(0) = 1$.

Euler's Method: An Algorithm & Error

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We decide on a step size h, and the x-values for the approximation are given by

$$x_n = x_0 + nh.$$

Euler's Method Formula: The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

with $h = 0.25$ starting w/ $x_0 = 0$,

we'll need 4 steps to get to $x_n = 1$.

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

$x_0 = 0$, $y_0 = 1$, $h = 0.25$, $f(x,y) = xy$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.25\,(0 \times 1) = 1$$

$x_1 = 0.25, \quad y_1 = 1, \quad h = 0.25$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1 + 0.25(0.25 \times 1) = 1.0625$

$x_2 = 0.5, \quad y_2 = 1.0625, \quad h = 0.25$

$y_3 = y_2 + hf(x_2, y_2)$

$= 1.0625 + 0.25(0.5 * 1.0625)$

$= 1.1953125$

$x_3 = 0.75, \quad y_3 = 1.1953125, \quad h = 0.25$

$$y_4 = y_3 + h f(x_3, y_3)$$

$$= 1.1953125 + 0.25\left(0.75 \times 1.1953125\right)$$

$$= 1.419434$$

$$y(1) \approx y_4 = 1.419434$$

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Taking a step size of $h = 0.25$, we went through this process and found that $y_4 = 1.41943$ was our approximation to $y(1)$.

The actual[1] solution value $y(1) = \sqrt{e} = 1.64872$. This raises the question of how good our approximation can be expected to be.

[1] The exact solution $y = e^{x^2/2}$.

Euler's Method: Error

As the previous examples suggest, the approximate solution obtained using Euler's method has error. Moreover, the error can be expected to become more pronounced, the farther away from the initial condition we get.

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[2]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

[2]Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We can ask, how does the error depend on the step size?

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different h values to approximate $y(1)$, and recorded the results shown in the table.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

$\boxed{\text{The absolute error in Euler's method is proportional to the step size.}}$

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- The error in approximating the curve with a tangent line, and
- using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Euler's method is an order 1 scheme.

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

Problem: Using a step size of $h = 0.2$, use Euler's method to approximate $x(1.4)$.

Here, $t_0 = 1$ and $x_0 = 2$. $f(t, x) = \dfrac{x^2 - t^2}{xt}$.

To get to $t_n = 1.4$ with $h = 0.2$, we need 2 steps.

$t_0 = 1$

$t_1 = 1.2$

$t_2 = 1.4$

$x(1.4) \approx x_2$

$t_0 = 1, \quad x_0 = 2, \quad h = 0.2$

$x_1 = x_0 + h f(t_0, x_0)$

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

$$x_1 = 2 + 0.2\left(\frac{2^2 - 1^2}{2(1)}\right) = 2.3$$

Now, $t_1 = 1.2$, $x_1 = 2.3$ $\quad h = 0.2$

$$x_2 = x_1 + h f(t_1, x_1)$$

$$= 2.3 + 0.2\left(\frac{2.3^2 - 1.2^2}{(2.3)(1.2)}\right)$$

$$= 2.579$$

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

We found $x(1.4) \approx 2.579$

The exact value turns out

to be $x(1.4) = 2.554$

It is possible to solve this IVP exactly to obtain the solution $x = \sqrt{4t^2 - 2t^2 \ln(t)}$. The true value $x(1.4) = 2.554$ to four decimal digits.

Existence and Uniqueness

Existence & Uniqueness Questions

Two important questions we can always pose (and sometimes answer) are

1. Does an IVP have a solution? (existence) and
2. If it does, is there just one? (uniqueness)

As a silly example, consider whether the following can be solved[3]

$$\underbrace{\left(\frac{dy}{dx}\right)^2 + 1}_{\geq 1} = \underbrace{-y^2}_{\leq 0}.$$

[3]If we only wish to consider real valued functions.

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Exercise 1: Verify that $y = \frac{x^4}{16}$ is a solution of the IVP.

Exercise 2: Can you find a second solution of the IVP by inspection—i.e. by clever guessing? (Hint: What's the simplest type of function you can think of. Is there one of that type that satisfies both the ODE and the initial condition?) Constant function $y=0$

This IVP has two distinct solutions. We'll see how to solve the ODE in the next section. The solution technique will give us a 1-parameter family of solutions. We'll find that one of the solutions is a member of the family, and one is not.