August 21 Math 2306 sec. 51 Spring 2023

Section 2: Initial Value Problems

Definition: Initial Value Problem

An **Initial Value Problem (IVP)** consists of a differential equation coupled with a certain type of additional conditions. For Example: Solve the equation [a]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem*.

[a] on some interval I containing x_0.

Note that y and its derivatives are evaluated at the same initial x value of x_0.

Examples for $n = 1$ or $n = 2$

First order case: $$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

The ODE part tells us about the shape of a solution curve, since the derivative is the slope. The initial condition indicates where the curve is. It has to pass through the point (x_0, y_0).

Second order case: $$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

The classic example of a second order IVP is motion of a particle. The ODE part is the acceleration information. In this case, y_0 tells us where the particle starts (initial position) and y_1 is the starting velocity.

Example

Given that $y = c_1 x + \dfrac{c_2}{x}$ is a 2-parameter family of solutions of the ODE $x^2y'' + xy' - y = 0$ on the interval $(0, \infty)$, solve the initial value problem

$$x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

First, we find all solutions to the ODE, then ensure the initial conditions are satisfied. We already know the solutions are $y = c_1 x + \frac{c_2}{x}$. So that we need to find the numbers c_1 and c_2 so that $y(1)=1$ and $y'(1)=3$.

$y' = c_1 - \frac{c_2}{x^2}$.

$y(1) = c_1(1) + \frac{c_2}{1} = 1 \Rightarrow c_1 + c_2 = 1$

$y'(1) = c_1 - \frac{c_2}{1^2} = 3 \Rightarrow c_1 - c_2 = 3$

Solve $\quad c_1 + c_2 = 1$

$$c_1 - c_2 = 3$$

add $\quad 2c_1 = 4 \Rightarrow c_1 = 2$

$$c_2 = 1 - c_1 = 1 - 2 = -1$$

$$\boxed{\text{The solution to the IVP is} \quad y = 2x - \frac{1}{x}}$$

Graphical Interpretation

The ODE $\dfrac{dy}{dx} = f(x, y)$ may give rise to many solution curves (a *family of solutions*). An initial condition requires the curve to pass through a certain point.

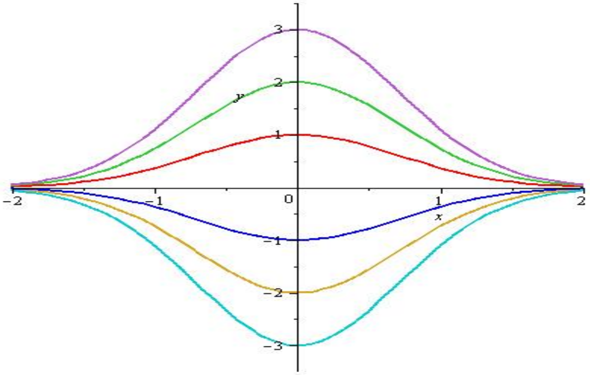

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

Example

The relation $y^2 - 2x^2y = C$ defines a 1-parameter family of solutions to the ODE $y' = \frac{2xy}{y-x^2}$.

Find an implicit solution to the initial value problem

$$\frac{dy}{dx} = \frac{2xy}{y-x^2}, \quad y(1) = -2.$$

The solutions are already given by $y^2 - 2x^2y = C$

The initial condition says that when $x=1$, $y=-2$.

substitute $(-2)^2 - 2(1)^2 \cdot (-2) = C$

$4 + 4 = C \Rightarrow C = 8$

An implicit solution to the IVP is

$$y^2 - 2x^2y = 8$$

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

In later sections, we'll have methods for solving some first order ODEs by hand. Here, we look at a method for *approximating* the solution called **Euler's Method**. The idea is simple

- Start with the point (x_0, y_0) that is given,
- use the ODE to make a tangent line $L(x)$ at (x_0, y_0),
- increment the independent variable to a new point x_1
- approximate the solution y using the tangent line, $y(x_1) \approx y_1 = L(x_1)$,
- rinse and repeat!

Euler's Method: $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

Let's go through an example, and then derive the general formula used for Euler's method.

For the next few slides, we will consider the example

$$\frac{dy}{dx} = xy, \quad \text{with initial condition} \quad y(0) = 1$$

Note that

$$f(x, y) = xy, \quad x_0 = 0, \quad \text{and} \quad y_0 = 1$$

We will build the solution in increments of 0.25. (This number is chosen for this example and can be changed.)

The true solution for this simple example is well know, so the true curve can be plotted along with the approximations. But keep in mind that, in general, the exact solution isn't known. (It it was, you wouldn't need to approximate it.)

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

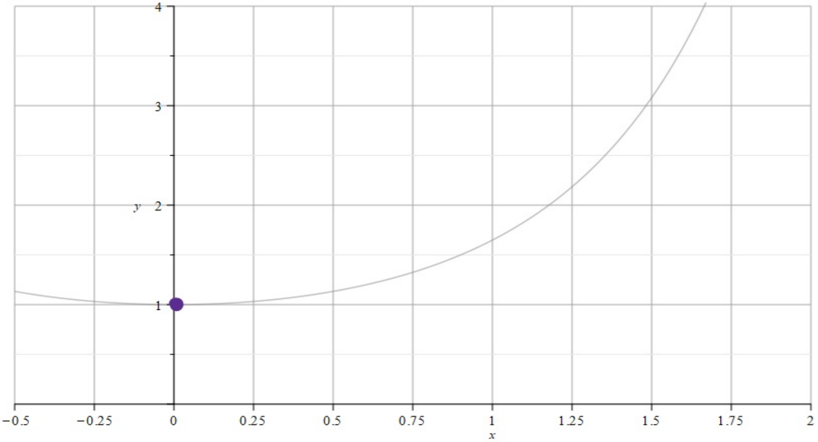

Figure: We know that the point $(x_0, y_0) = (0, 1)$ is on the curve. And the slope of the curve at $(0, 1)$ is $m_0 = f(0, 1) = 0 \cdot 1 = 0$.
Note: The gray curve is the true solution to this IVP. It's shown for reference.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

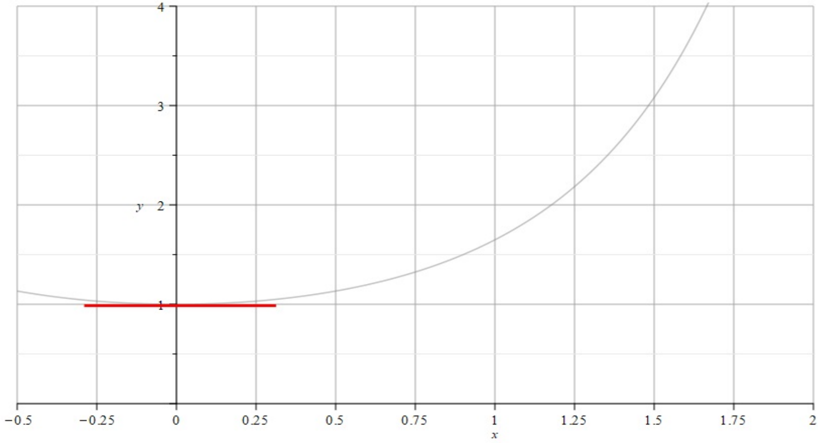

Figure: So we draw a little tangent line (we know the point and slope). Then we increase x, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line y_1. So $y_1 \approx y(x_1)$. (I'm taking $h = 0.25$.)

Example $\frac{dy}{dx} = xy, \quad y(0) = 1$

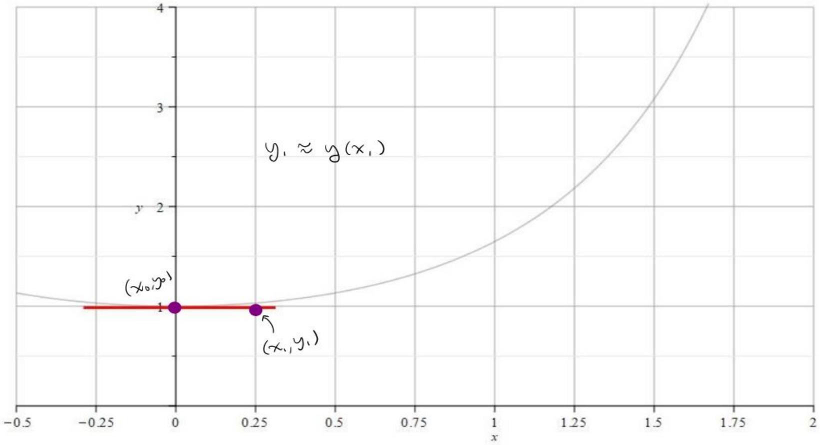

Figure: We take the approximation to the true function y at the point $x_1 = x_0 + h$ to be the point on the tangent line.

Example $\frac{dy}{dx} = xy, \quad y(0) = 1$

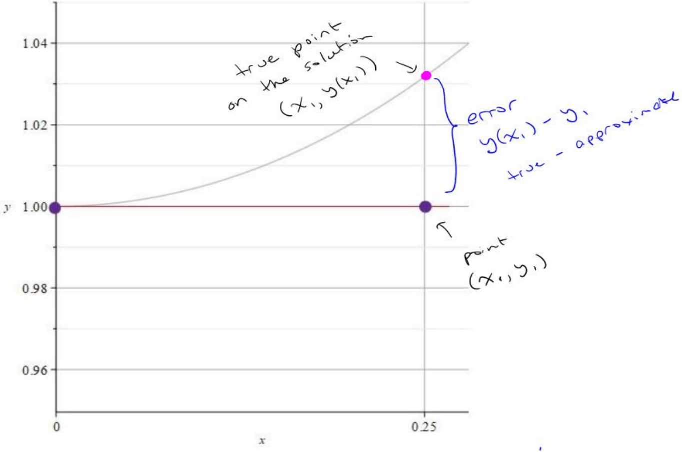

Figure: When h is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact y value and the approximation from the tangent line.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

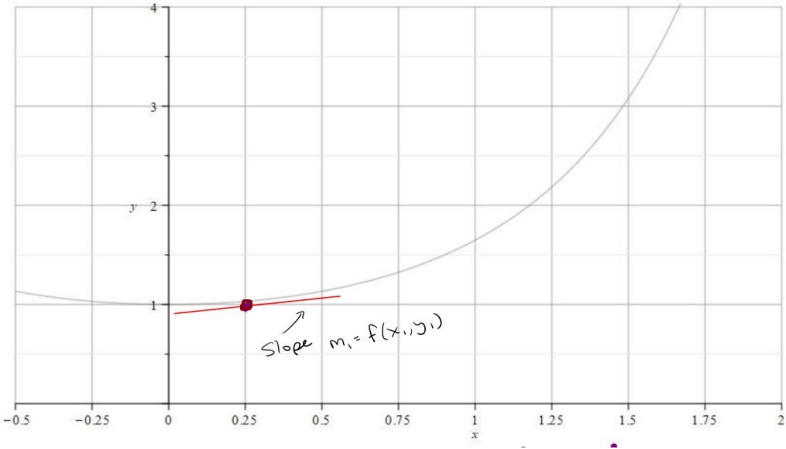

Figure: Now we start with the point (x_1, y_1) and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through (x_1, y_1) with slope m_1.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

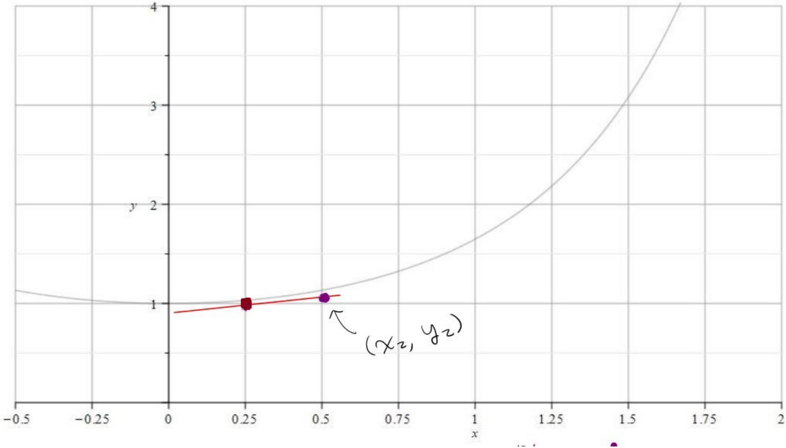

Figure: We go out h more units to $x_2 = x_1 + h$. Pick the point on the tangent line (x_2, y_2), and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

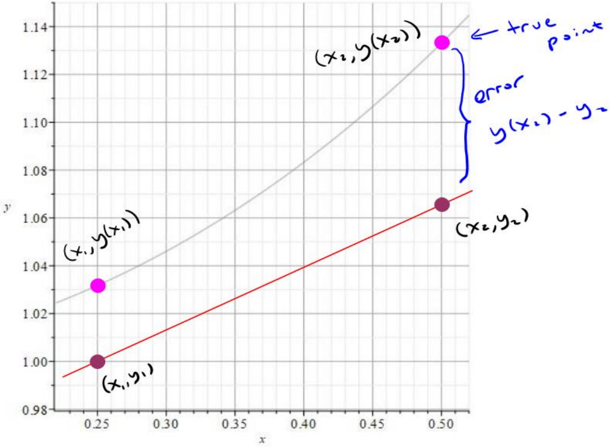

Figure: If we zoom in, we can see that there is some error. But as long as h is small, the point on the tangent line approximates the point on the actual solution curve.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

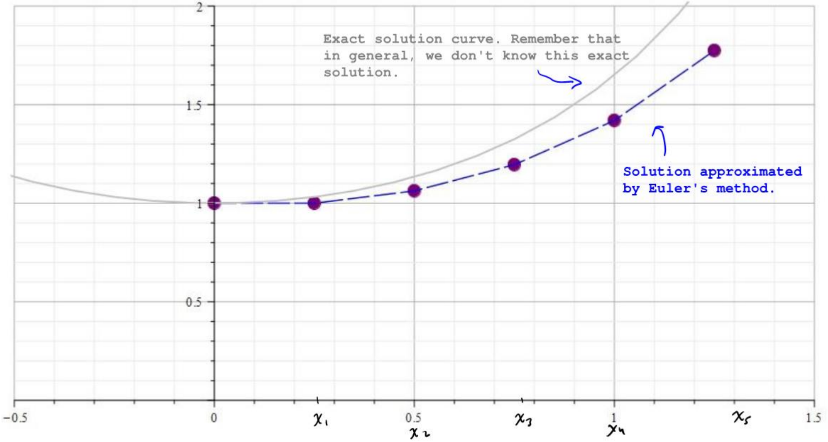

Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution y

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_N, y_N).$$

We'll take the x values to be equally spaced with a common difference of h. That is

$$\begin{aligned}
x_1 &= x_0 + h \\
x_2 &= x_1 + h = x_0 + 2h \\
x_3 &= x_2 + h = x_0 + 3h \\
&\vdots \\
x_n &= x_0 + nh
\end{aligned}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Notation:

- y_n will denote our approximation, and
- $y(x_n)$ will denote the exact solution (that we don't know)

To build a formula for the approximation y_1, let's approximate the derivative at (x_0, y_0).

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope.)

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Let's get a formula for y_1.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0) \quad , \quad x_1 - x_0 = h$$

$$\frac{y_1 - y_0}{h} = f(x_0, y_0)$$

$$y_1 - y_0 = h f(x_0, y_0)$$

$$\boxed{y_1 = y_0 + h f(x_0, y_0)}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \Longrightarrow \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

Euler's Method Formula: The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.50, \quad x_3 = 0.75, \quad x_4 = 1$

$f(x,y) = xy, \quad x_0 = 0, \quad y_0 = 1 \qquad h = 0.25$

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.25(0 \cdot 1) = 1$

$x_1 = 0.25, \quad y_1 = 1$

$y_2 = y_1 + hf(x_1, y_1) = 1 + 0.25(0.25 \cdot 1)$

$= 1.0625$

$x_2 = 0.5 \quad , \quad y_2 = 1.0625$

$$y_3 = y_2 + h\, f(x_2, y_2)$$

$$= 1.0625 + 0.25\,(0.5 \cdot 1.0625)$$

$$= 1.19531$$

$x_3 = 0.75 \qquad y_3 = 1.19531$

$$y_4 = y_3 + h\, f(x_3, y_3)$$

$$= 1.19531 + 0.25\,(0.75 \cdot 1.19531)$$

$$= 1.41943$$

We'll compare this to the exact solution next time.