# August 21 Math 2306 sec. 52 Spring 2023

## Section 2: Initial Value Problems

### Definition: Initial Value Problem

An **Initial Value Problem (IVP)** consists of a differential equation coupled with a certain type of additional conditions. For Example: Solve the equation [a]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem*.

---

[a] on some interval $I$ containing $x_0$.

**Note** that $y$ and its derivatives are evaluated at the same initial $x$ value of $x_0$.

# Examples for $n = 1$ or $n = 2$

First order case: $$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

The ODE tells us about the shape of any solution curve. since $\frac{dy}{dx}$ is slope. The initial condition specifies where the curve is in space. It passes through $(x_0, y_0)$.

Second order case: $$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

If $y$ is the position of a particle, the ODE tells us about the acceleration.

$y_0$ is the initial position and

$y_1$ is the initial velocity.

# Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of the ODE $x^2 y'' + xy' - y = 0$ on the interval $(0, \infty)$, solve the initial value problem

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

In general, we find all solutions to the ODE, then find the one that satisfies $y(1)=1$ and $y'(1)=3$.

We already know that all solutions are functions $y = c_1 x + \frac{c_2}{x}$. We have to find the numbers $c_1$ and $c_2$ that satisfy the initial conditions.

$$y = c_1 x + \frac{c_2}{x}, \quad y' = c_1 - \frac{c_2}{x^2}$$

$$y(1) = c_1(1) + \frac{c_2}{1} = 1 \qquad y'(1) = c_1 - \frac{c_2}{1^2} = 3$$

$\Rightarrow \quad c_1 + c_2 = 1 \quad$ and $\quad c_1 - c_2 = 3$

Solve this system.

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3 \\ \hline \text{add} \quad 2c_1 &= 4 \quad \Rightarrow \quad c_1 = 2 \end{aligned}$$

$$c_2 = 1 - c_1 = 1 - 2 = -1$$

The solution to the IVP is

$$y = 2x - \frac{1}{x}.$$

# Graphical Interpretation

The ODE $\dfrac{dy}{dx} = f(x, y)$ may give rise to many solution curves (a *family of solutions*). An initial condition requires the curve to pass through a certain point.

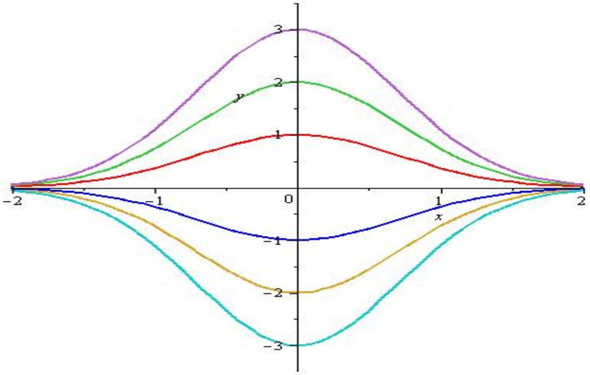


Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of $y_0$

# Example

The relation $y^2 - 2x^2y = C$ defines a 1-parameter family of solutions to the ODE $y' = \frac{2xy}{y-x^2}$.

Find an implicit solution to the initial value problem

$$\frac{dy}{dx} = \frac{2xy}{y - x^2}, \quad y(1) = -2.$$

We're given that the solutions are defined by $y^2 - 2x^2y = C$. We need $C$ such that $y(1) = -2$. When $x = 1$, $y = -2$

$$(-2)^2 - 2(1)^2(-2) = C$$

$$4 + 4 = 8 \implies C = 8$$

The solution is defined by

$$y^2 - 2x^2y = 8$$

# A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

In later sections, we'll have methods for solving some first order ODEs by hand. Here, we look at a method for *approximating* the solution called **Euler's Method**. The idea is simple

- Start with the point $(x_0, y_0)$ that is given,
- use the ODE to make a tangent line $L(x)$ at $(x_0, y_0)$,
- increment the independent variable to a new point $x_1$
- approximate the solution $y$ using the tangent line, $y(x_1) \approx y_1 = L(x_1)$,
- rinse and repeat!

# Euler's Method: $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

Let's go through an example, and then derive the general formula used for Euler's method.

For the next few slides, we will consider the example

$$\frac{dy}{dx} = xy, \quad \text{with initial condition} \quad y(0) = 1$$

Note that

$$f(x, y) = xy, \quad x_0 = 0, \quad \text{and} \quad y_0 = 1$$

We will build the solution in increments of 0.25. (This number is chosen for this example and can be changed.)

The true solution for this simple example is well know, so the true curve can be plotted along with the approximations. But keep in mind that, in general, the exact solution isn't known. (It it was, you wouldn't need to approximate it.)

## Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

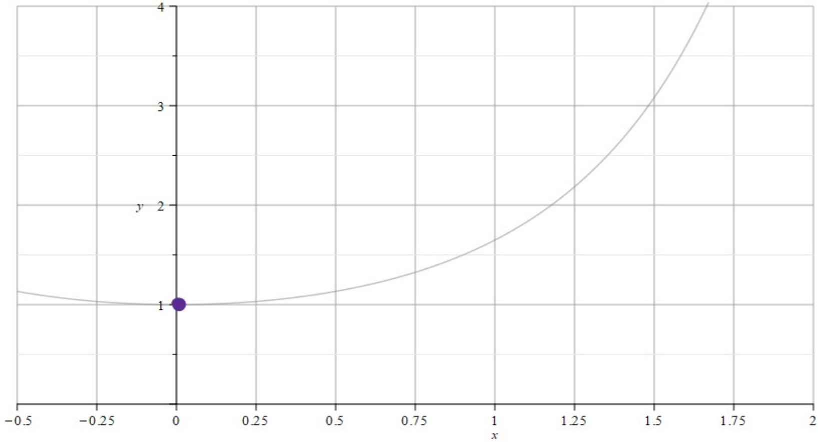


Figure: We know that the point $(x_0, y_0) = (0, 1)$ is on the curve. And the slope of the curve at $(0, 1)$ is $m_0 = f(0, 1) = 0 \cdot 1 = 0$.
Note: The gray curve is the true solution to this IVP. It's shown for reference.

# Example $\frac{dy}{dx} = xy, \quad y(0) = 1$

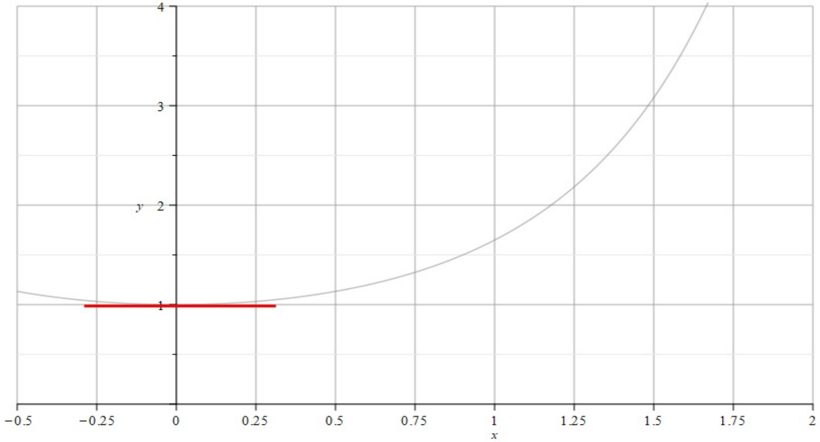


Figure: So we draw a little tangent line (we know the point and slope). Then we increase $x$, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line $y_1$. So $y_1 \approx y(x_1)$. (I'm taking $h = 0.25$.)

## Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

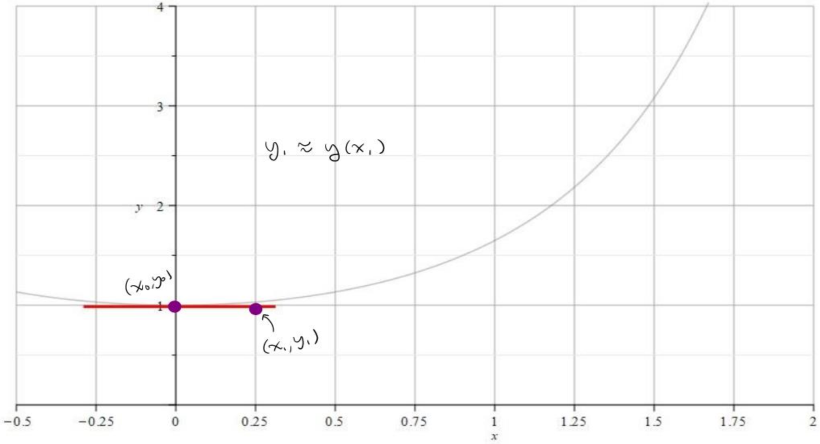


Figure: We take the approximation to the true function $y$ at the point $x_1 = x_0 + h$ to be the point on the tangent line.

# Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

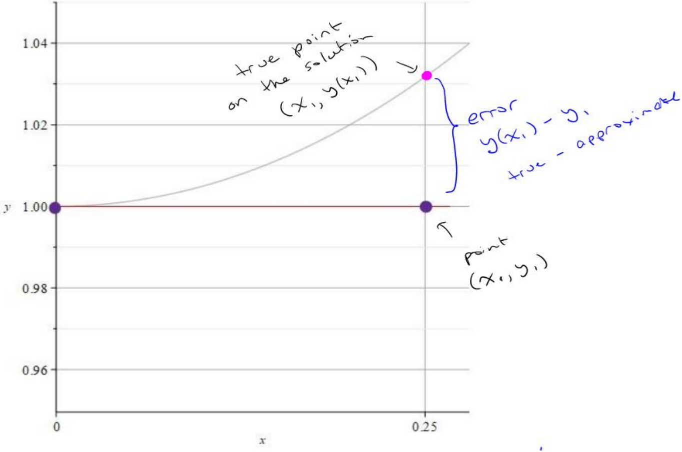


Figure: When $h$ is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact $y$ value and the approximation from the tangent line.

## Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

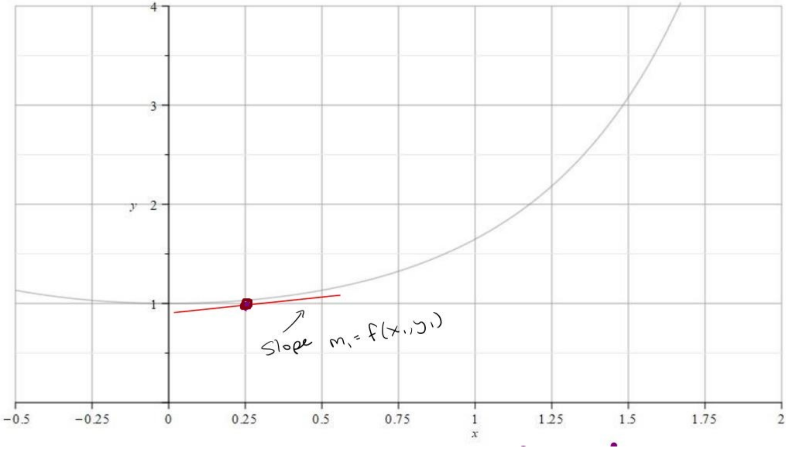


Figure: Now we start with the point $(x_1, y_1)$ and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through $(x_1, y_1)$ with slope $m_1$.

# Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

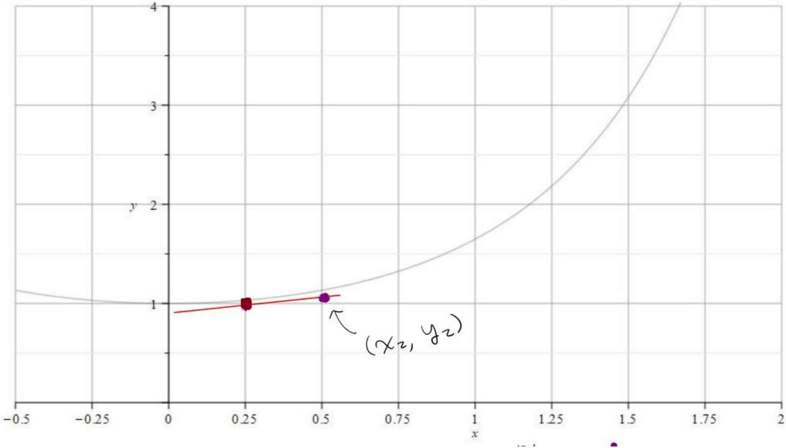


Figure: We go out $h$ more units to $x_2 = x_1 + h$. Pick the point on the tangent line $(x_2, y_2)$, and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

# Example $\frac{dy}{dx} = xy, \quad y(0) = 1$

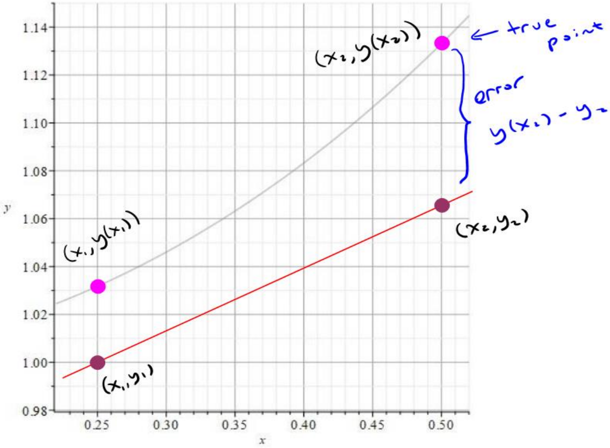


Figure: If we zoom in, we can see that there is some error. But as long as $h$ is small, the point on the tangent line approximates the point on the actual solution curve.

# Example $\frac{dy}{dx} = xy, \quad y(0) = 1$

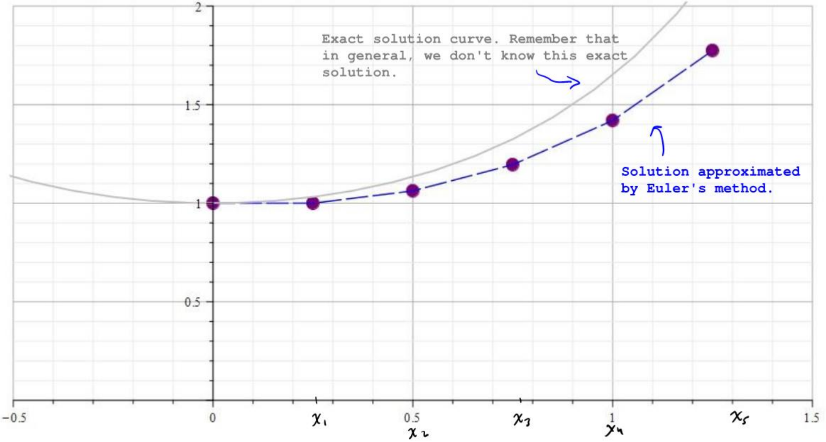


Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

# Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution $y$

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_N, y_N).$$

We'll take the $x$ values to be equally spaced with a common difference of $h$. That is

$$\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_1 + h = x_0 + 2h \\ x_3 &= x_2 + h = x_0 + 3h \\ &\vdots \\ x_n &= x_0 + nh \end{aligned}$$

# Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

**Notation:**

- $y_n$ will denote our approximation, and
- $y(x_n)$ will denote the exact solution (that we don't know)

To build a formula for the approximation $y_1$, let's approximate the derivative at $(x_0, y_0)$.

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope. )

# Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Let's get a formula for $y_1$.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0), \quad x_1 - x_0 = h$$

$$\frac{y_1 - y_0}{h} = f(x_0, y_0)$$

$$y_1 - y_0 = h\, f(x_0, y_0)$$

$$\Rightarrow \boxed{y_1 = y_0 + h\, f(x_0, y_0)}$$

# Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \Longrightarrow \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

**Euler's Method Formula:** The $n^{th}$ approximation $y_n$ to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with $(x_0, y_0)$ given in the original IVP and $h$ the choice of step size.

# Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$f(x,y) = xy$, $x_0 = 0$, $y_0 = 1$, $h = 0.25$

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.25(0 \cdot 1) = 1$

$x_1 = 0.25$, $y_1 = 1$

$y_2 = y_1 + hf(x_1, y_1) = 1 + 0.25(0.25 \cdot 1)$

$= 1.0625$

$x_2 = 0.5$ $y_2 = 1.0625$

$$y_3 = y_2 + h\, f(x_2, y_2)$$

$$= 1.0625 + 0.25\,(0.5 \cdot 1.0625)$$

$$= 1.19531$$

$$x_3 = 0.75, \quad y_3 = 1.19531$$

$$y_4 = y_3 + h\, f(x_3, y_3)$$

$$= 1.19531 + 0.25\,(0.75 \cdot 1.19531)$$

$$= 1.41943$$

$$y_4 \approx y(1)$$

# Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Taking a step size of $h = 0.25$, we went through this process and found that $y_4 = 1.41943$ was our approximation to $y(1)$.

The actual[1] solution value $y(1) = \sqrt{e} = 1.64872$. This raises the question of how good our approximation can be expected to be.

[1] The exact solution $y = e^{x^2/2}$.