Section 2: Initial Value Problems

Recall that an **initial value problem (IVP)** consists of a differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

coupled with a set if **initial conditions (IC)**

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}.$$

Recall the key features

- The number of IC matches the order of the ODE, and
- all IC are given at the same specified input value x_0.

IVPs

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Third order case:

$$\frac{d^3y}{dx^3} = f(x, y, y', y''), \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2$$

and so forth

System of IVPs

We can also consider a system of IVP for example

$$
\begin{aligned}
\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60, \quad i_2(t_0) = i_{2_0} \\
\frac{di_3}{dt} &= -2i_2 - 5i_3 + 60, \quad i_3(t_0) = i_{3_0}
\end{aligned}
$$

The number of initial conditions for each dependent variable will match the highest order derivative for *that* dependent variable. All initial conditions for all dependent variables are given at the same input value (t_0).

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2y'' + xy' - y = 0$, solve the IVP

$$x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

The solutions to the ODE are known. We need to determine the values for c_1 and c_2 that satisfy the IC.

$$y = c_1 x + \frac{c_2}{x} \qquad y(1) = c_1(1) + \frac{c_2}{1} = 1$$

$$y' = c_1 - \frac{c_2}{x^2} \qquad y'(1) = c_1 - \frac{c_2}{1^2} = 3$$

We need to solve the system

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 3$$

add

$$2c_1 = 4$$
$$\Rightarrow c_1 = 2$$

From the 1st equation, $c_2 = 1 - c_1 = 1 - 2 = -1$

$$\boxed{\text{The solution to the IVP is } \quad y = 2x - \frac{1}{x}}$$

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Euler's Method is a scheme for finding an approximate solution to this IVP. The basic idea is that we

- Start with the known point (x_0, y_0) on the solution curve,
- use the slope (given by $\frac{dy}{dx}$) to get a tangent line there, and
- approximate a nearby point on the curve by the tangent line.
- march forward a littel bit, and repeat.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

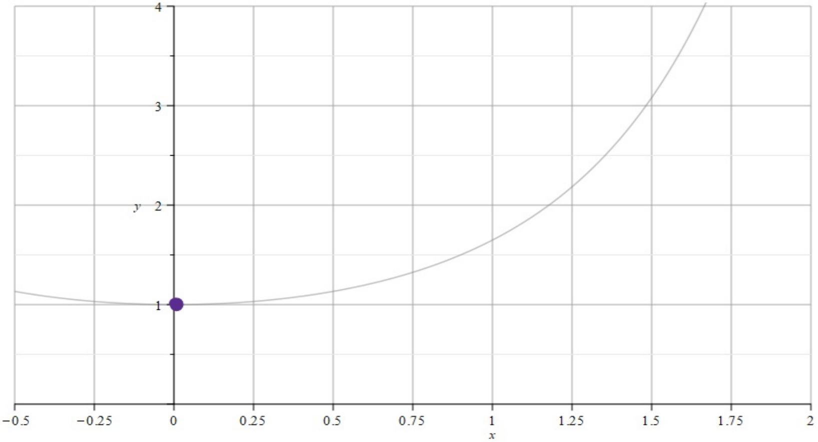

Figure: We know that the point $(x_0, y_0) = (0, 1)$ is on the curve. And the slope of the curve at $(0, 1)$ is $m_0 = f(0, 1) = 0 \cdot 1 = 0$.
Note: The gray curve is the true solution to this IVP. It's shown for reference.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

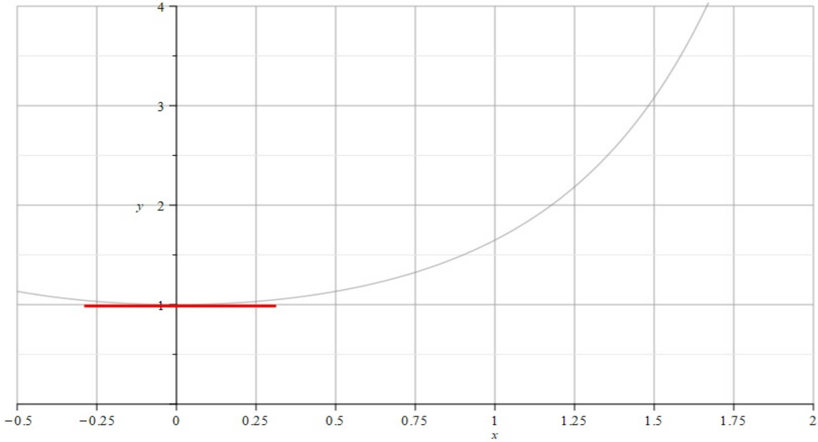

Figure: So we draw a little tangent line (we know the point and slope). Then we increase x, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line y_1. So $y_1 \approx y(x_1)$.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

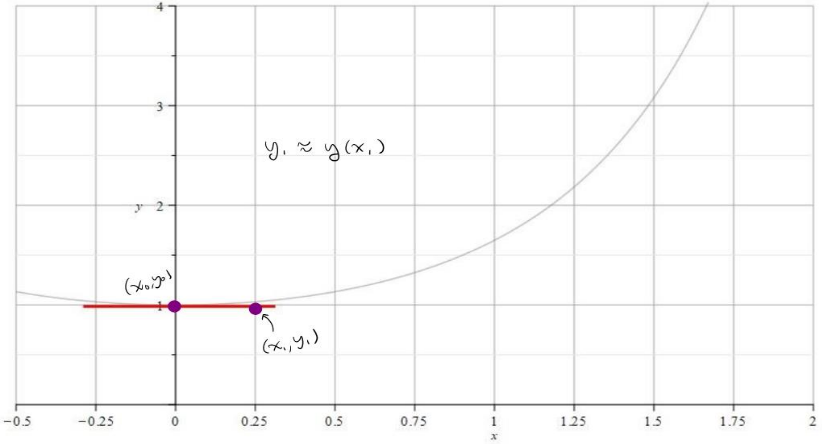

Figure: We take the approximation to the true function y at the point $x_1 = x_0 + h$ to be the point on the tangent line.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

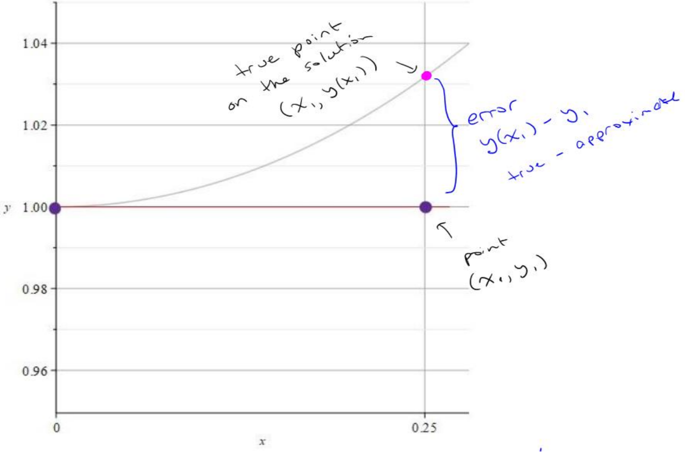

Figure: When h is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact y value and the approximation from the tangent line.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

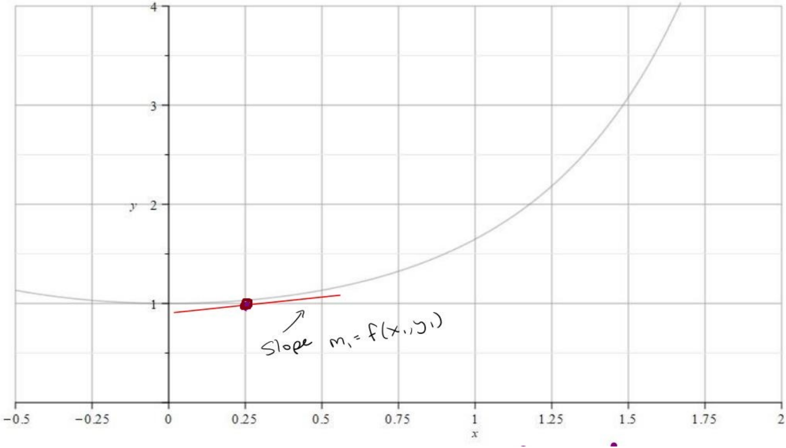

Figure: Now we start with the point (x_1, y_1) and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through (x_1, y_1) with slope m_1.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

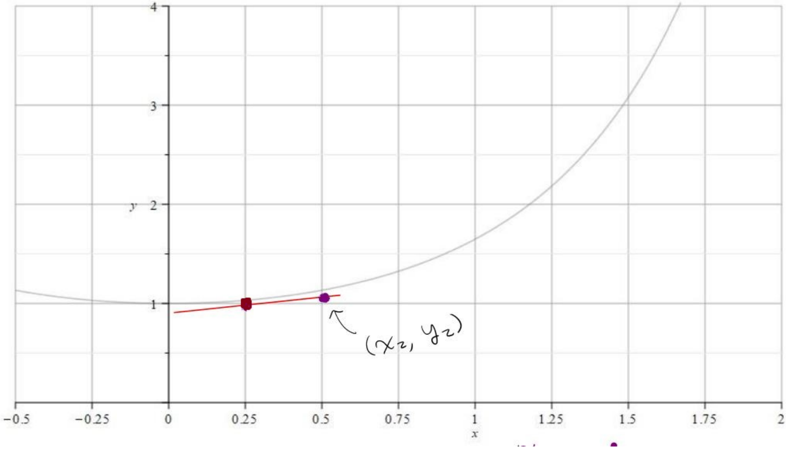

Figure: We go out h more units to $x_2 = x_1 + h$. Pick the point on the tangent line (x_2, y_2), and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

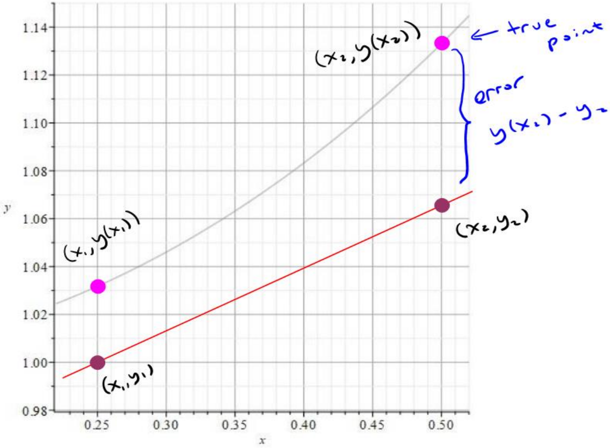

Figure: If we zoom in, we can see that there is some error. But as long as h is small, the point on the tangent line approximates the point on the actual solution curve.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

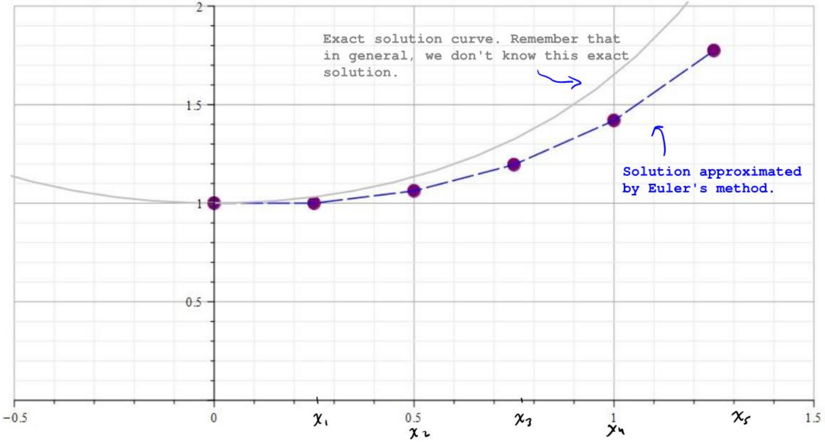

Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution y

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_N, y_N).$$

We'll take the x values to be equally spaced with a common difference of h. That is

$$\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_1 + h = x_0 + 2h \\ x_3 &= x_2 + h = x_0 + 3h \\ &\vdots \\ x_n &= x_0 + nh \end{aligned}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Notation:

- y_n will denote our approximation, and
- $y(x_n)$ will denote the exact solution (that we don't know)

To build a formula for the approximation y_1, let's approximate the derivative at (x_0, y_0).

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope.)

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Let's get a formula for y_1.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0) \quad , \quad x_1 - x_0 = h$$

$$\frac{y_1 - y_0}{h} = f(x_0, y_0) \implies y_1 - y_0 = h f(x_0, y_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \Longrightarrow \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

Euler's Method Formula: The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

with $x_0 = 0$ and $h = 0.25$,

$x_1 = 0.25$, $x_2 = 0.50$, $x_3 = 0.75$, $x_4 = 1.0$

$x_0 = 0$, $y_0 = 1$, $h = 0.25$, $f(x,y) = xy$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.25(0 \cdot 1) = 1$$

$x_1 = 0.25$, $y_1 = 1$, $h = 0.25$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 1 + 0.25\,(0.25 \cdot 1) = 1.0625$$

$$x_2 = 0.5 \qquad y_2 = 1.0625 \qquad h = 0.25$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 1.0625 + 0.25\,(0.5 \cdot 1.0625)$$

$$= 1.19531$$

$$x_3 = 0.75, \quad y_3 = 1.19531, \quad h = 0.25$$

$$y_4 = y_3 + h f(x_3, y_3)$$

$$= 1.19531 + 0.25\left(0.75 \cdot 1.19531\right)$$

$$= 1.41943$$

$$y_4 = 1.41943 \approx y(x_4) = y(1)$$

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

We went through this process and found that $y_4 = 1.41943$ was our approximation to $y(1)$.

The true[1] $y(1) = \sqrt{e} = 1.64872$. This raises the question of how good our approximation can be expected to be.

[1]The exact solution $y = e^{x^2/2}$.

Euler's Method: Error

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[2]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

[2] Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We can ask, how does the error depend on the step size?

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different h values to approximate $y(1)$, and recorded the results shown in the table.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

$$\boxed{\text{The absolute error in Euler's method is proportional to the step size.}}$$

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- The error in approximating the curve with a tangent line, and
- using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Euler's method is an order 1 scheme.