August 22 Math 3260 sec. 53 Fall 2025

Section 1.1 The Vector Space R^2

- We have defined vectors in R^2 and are working with scalars in R.
- We defined **vector addition** $\langle x_1, x_2\rangle + \langle y_1, y_2\rangle = \langle x_1 + y_1, x_2 + y_2\rangle$,
- and **scalar multiplication** $c\langle x_1, x_2\rangle = \langle cx_1, cx_2\rangle$.
- The **zero vector**, $\vec{0}_2$, in R^2 is the additive identity, and
- each vector $\vec{x}$ has an **additive inverse** vector, $-\vec{x}$.
- Given a collection of vectors, $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$, a **linear combination** is any vector of the form $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k$ where $c_1, \dots, c_k$ are scalars.

Section 1.1 The Vector Space R^2

- We defined the **magnitude** of a vector $\|\vec{x}\| = \|\langle x_1, x_2\rangle\| = \sqrt{x_1^2 + x_2^2}$.
- A **unit vector** is a vector of magnitude 1.
- Two nonzero vectors $\vec{x}$ and $\vec{y}$ are **parallel** if and only if there exists a scalar c such that $\vec{y} = c\vec{x}$.
- The previous point guarantees that given any nonzero vector $\vec{x}$, we can find a unit vector parallel to $\vec{x}$.

Now, we want to come up with a characterization for vectors in R^2 that are perpendicular.

Perpendicular Vectors

We would like to arrive at a characterization for vectors that are perpendicular. By perpendicular, we mean that the standard representations of the vectors form a right angle.

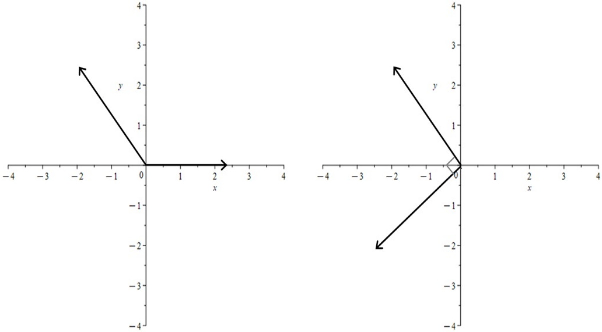

Figure: What should be true about nonzero vectors $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$ if they make an angle of 90°?

Perpendicular

We can look at the parallelogram determined by two vectors $\vec{x}$ and $\vec{y}$.

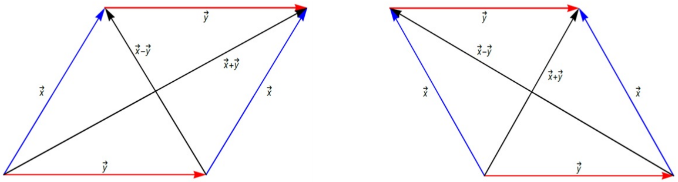

Figure: If the angle between the vectors is acute, then $\|\vec{x}+\vec{y}\| > \|\vec{x}-\vec{y}\|$, and if the angle between the vectors is obtuse, then $\|\vec{x}+\vec{y}\| < \|\vec{x}-\vec{y}\|$.

If the vectors are perpendicular, then the parallelogram will be a rectangle. A rectangle has diagonals of equal length.

Perpendicular

The diagonals of the parallelogram have lengths $\|\vec{x}+\vec{y}\|$ and $\|\vec{x}-\vec{y}\|$.

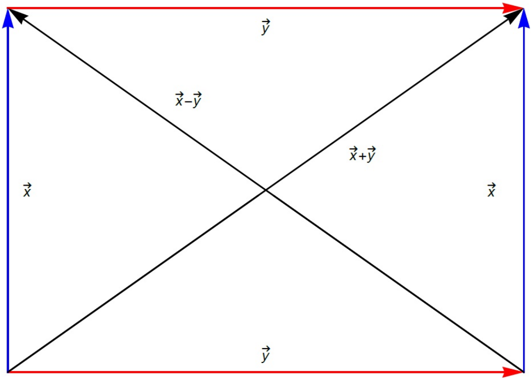

Figure: If the angle between the vectors 90°, then $\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|$.

Perpendicular

The nonzero vectors $\vec{x}$ and $\vec{y}$ are perpendicular if and only if

$$\|\vec{x} + \vec{y}\| = \|\vec{x} - \vec{y}\|.$$

Theorem: The nonzero vectors $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$ are perpendicular if and only if

$$x_1 y_1 + x_2 y_2 = 0.$$

Proof

Let's prove that $\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|$ is equivalent to $x_1y_1 + x_2y_2 = 0$.

We have to show that $\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|$ implies $x_1y_1 + x_2y_2 = 0$, and vice versa. Note that

$$\|\vec{x}+\vec{y}\|^2 = \| \langle x_1+y_1, x_2+y_2 \rangle \|^2$$

$$= (x_1+y_1)^2 + (x_2+y_2)^2$$

$$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2$$

$$\|\vec{x}+\vec{y}\|^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2(x_1y_1 + x_2y_2)$$

$$\|\vec{x}-\vec{y}\|^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2(x_1y_1 + x_2y_2)$$

Suppose $x_1y_1 + x_2y_2 = 0$. Then

$$\|\vec{x}+\vec{y}\|^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = \|\vec{x}-\vec{y}\|^2.$$ And since

$\|\vec{x} \pm \vec{y}\|$ are nonnegative,

$$\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|.$$

Conversely, suppose $\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|$.

Then $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}-\vec{y}\|^2$, and

$$x_1^2+y_1^2+x_2^2+y_2^2+2(x_1y_1+x_2y_2) = x_1^2+y_1^2+x_2^2+y_2^2-2(x_1y_1+x_2y_2)$$

$$\Rightarrow \quad 2(x_1y_1+x_2y_2) = -2(x_1y_1+x_2y_2)$$

$$\Rightarrow \quad 4(x_1y_1+x_2y_2) = 0$$

$$\Longrightarrow \quad x_1y_1 + x_2y_2 = 0$$

We've shown that if $x_1y_1 + x_2y_2 = 0$, then $\|\vec{x} + \vec{y}\| = \|\vec{x} - \vec{y}\|$. And we've shown that if $\|\vec{x} + \vec{y}\| = \|\vec{x} - \vec{y}\|$, then

$$x_1y_1 + x_2y_2 = 0.$$

Dot Product

The product $x_1y_1 + x_2y_2$ is significant. This represents a new operation on R^2.

The Dot Product

Given the pair of vectors $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$ in R^2, the **dot product** of $\vec{x}$ and $\vec{y}$, denoted

$$\vec{x} \cdot \vec{y},$$

is given by

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2.$$

Remark: The dot product of a pair of vectors in a **scalar**[1].

Remark: Two nonzero vectors are perpendicular if their dot product is zero.

[1] The dot product is an example of something called a *scalar* or an *inner* product.

Example: Determine whether the given vectors are parallel, perpendicular or neither.

1. $\vec{u} = \langle 2, 3 \rangle$ and $\vec{v} = \langle 6, -4 \rangle$

Are they parallel? is $\vec{u} = c\vec{v}$ for some c?

No, they're not parallel.

Are they perpendicular? is $\vec{u} \cdot \vec{v} = 0$?

$\vec{u} \cdot \vec{v} = 2(6) + 3(-4) = 12 - 12 = 0$

They are perpendicular.

2. $\vec{z} = \langle 1, -3 \rangle$ and $\vec{w} = \langle -2, 6 \rangle$

Are they parallel? $\vec{z} = c\vec{w}$, $\vec{z} = -\frac{1}{2}\vec{w}$ so

they are parallel.

Example: Determine whether the given vectors are parallel, perpendicular or neither.

3. $\vec{x} = \langle 1, -2 \rangle$ and $\vec{y} = \langle 2, 2 \rangle$

Parallel? $\vec{x} = c\vec{y}$?

Perpendicular? $\vec{x} \cdot \vec{y} = 1(2) + (-2)(2) = -2$.

They're neither parallel nor perpendicular.

4. $\vec{p} = \langle a, -b \rangle$ and $\vec{q} = \langle b, a \rangle$, where a, b are real numbers (not both zero)

$\vec{p} \cdot \vec{q} = a(b) + (-b)a = 0$ They're perpendicular.

Could they be parallel?

$\vec{p} = c\vec{q} \Rightarrow a = cb$ and $-b = ca$, $a \neq 0$ because they're not both zero. And $a = cb = c(-ca) = -c^2 a$

$\Rightarrow c^2 = -1$ which isn't possible.

Orthogonality

We should observe that for any vector $\vec{x} = \langle x_1, x_2 \rangle$ in R^2, the dot product

$$\vec{x} \cdot \vec{0}_2 = 0.$$

But the zero vector doesn't define an angle with $\vec{x}$. We have a generalization of the notion of perpendicularity.

Orthogonality

We say that two vectors $\vec{x}$ and $\vec{y}$ in R^2 are **orthogonal** if

$$\vec{x} \cdot \vec{y} = 0.$$