August 23 Math 2306 sec. 51 Fall 2021

Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

Solve the equation [1]

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).

[1] on some interval I containing x_0.

IVPs

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

1st order ODE

plus one condition

The initial condition tells us that the point (x_0, y_0) is on the graph of the solution y.

IVPs

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

2nd order ODE

two initial conditions

For example, if y is the position of a particle moving along a line at the time x, the ODE tells us about the acceleration, y_0 is the initial position, and y_1 is the initial velocity.

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

We have the solutions to the ODE part, $y = c_1 x + \frac{c_2}{x}$. We have to find numbers c_1 and c_2 to satisfy the initial conditions (IC).

$$y = c_1 x + \frac{c_2}{x}, \quad y' = c_1 - \frac{c_2}{x^2}$$

$$y(1) = 1 \Rightarrow 1 = c_1(1) + \frac{c_2}{1} \Rightarrow c_1 + c_2 = 1$$

$y'(1) = 3 \Rightarrow \quad 3 = c_1 - \frac{c_2}{1^2} \Rightarrow \quad c_1 - c_2 = 3$

we need to solve the system

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3 \end{aligned} \quad \text{add eqns} \quad \begin{aligned} 2c_1 &= 4 \\ c_1 &= 2 \\ c_2 &= -1 \end{aligned}$$

The solution to the IVP is

$$\boxed{y = 2x - \frac{1}{x}}$$

Graphical Interpretation

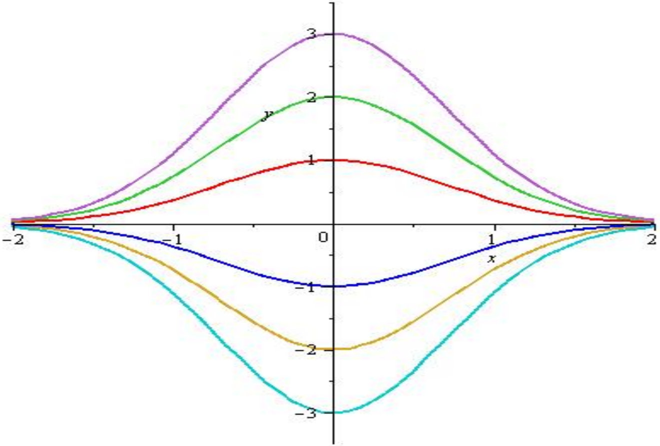

Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Euler's Method is a scheme for finding an approximate solution to this IVP. The basic idea is that we

- Start with the known point (x_0, y_0) on the solution curve,
- use the slope (given by $\frac{dy}{dx}$) to get a tangent line there, and
- approximate a nearby point on the curve by the tangent line.
- march forward a littel bit, and repeat.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

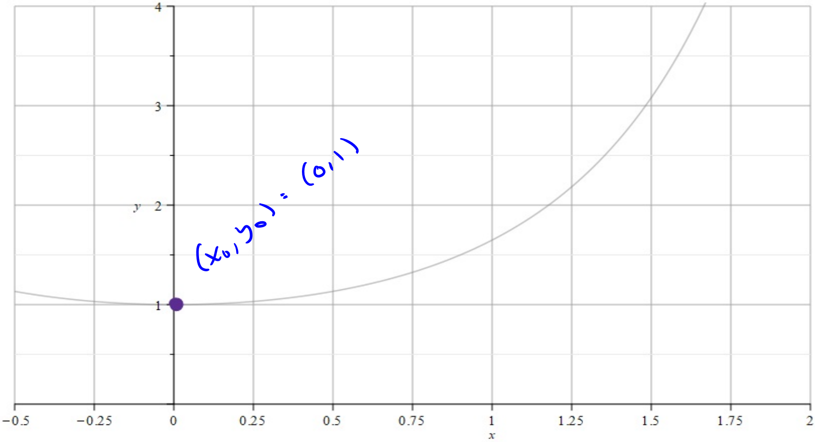

Figure: We know that the point $(x_0, y_0) = (0,1)$ is on the curve. And the slope of the curve at $(0,1)$ is $m_0 = f(0,1) = 0 \cdot 1 = 0$.
Note: The gray curve is the true solution to this IVP. It's shown for reference.

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

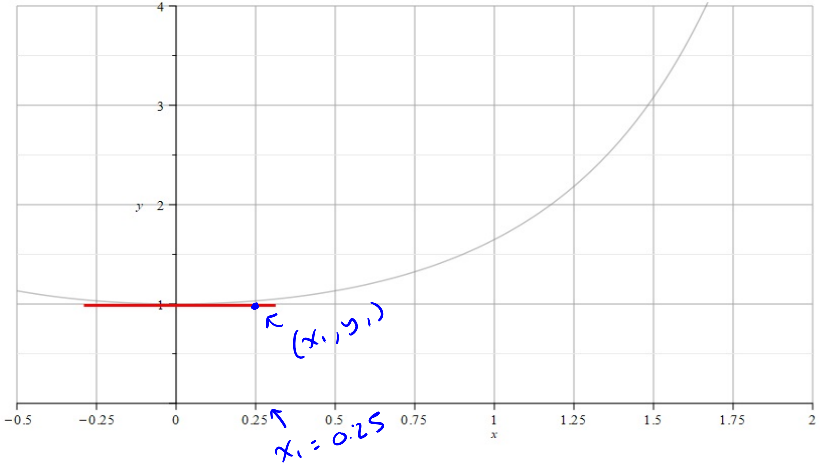

Figure: So we draw a little tangent line (we know the point and slope). Then we increase x, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line y_1. So $y_1 \approx y(x_1)$.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

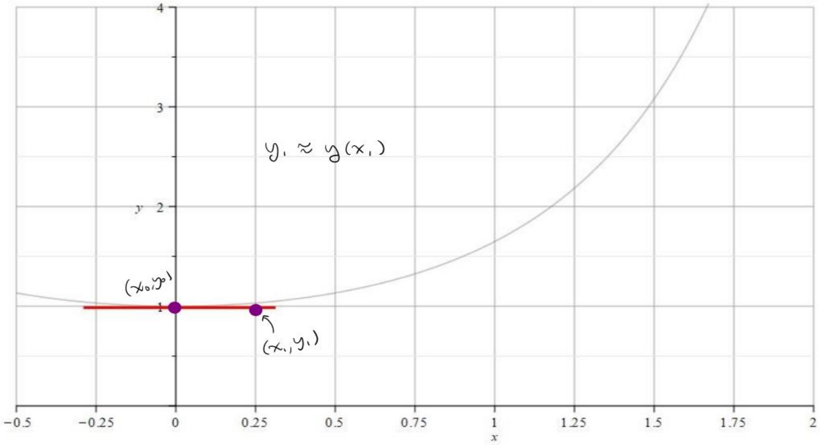

Figure: We take the approximation to the true function y at the point $x_1 = x_0 + h$ to be the point on the tangent line.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

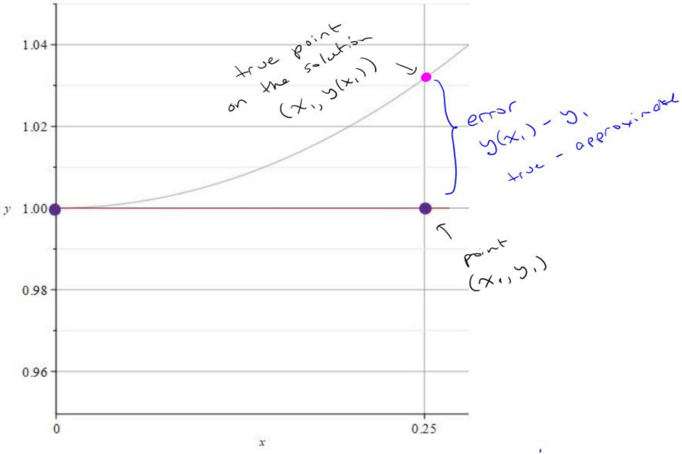

Figure: When h is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact y value and the approximation from the tangent line.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

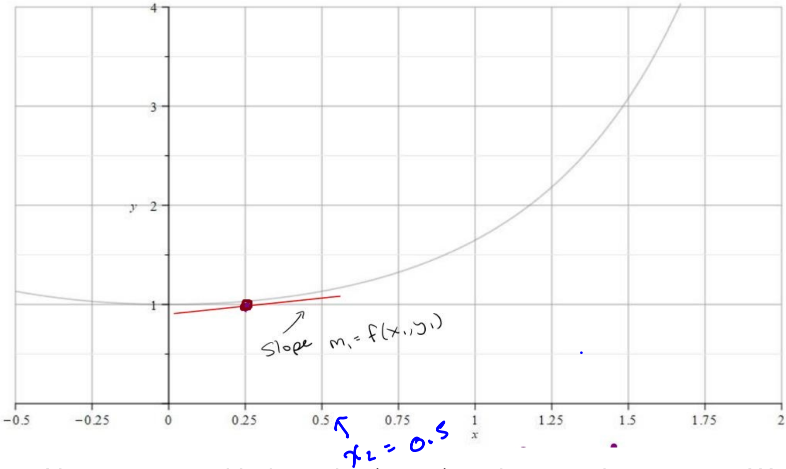

Figure: Now we start with the point (x_1, y_1) and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through (x_1, y_1) with slope m_1.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

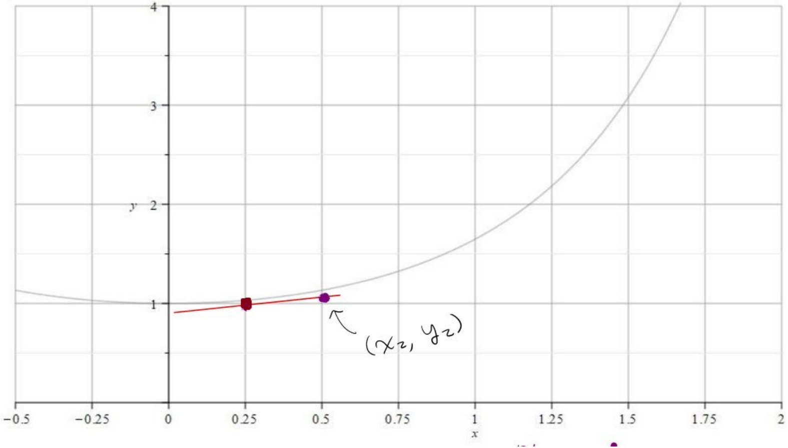

Figure: We go out h more units to $x_2 = x_1 + h$. Pick the point on the tangent line (x_2, y_2), and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

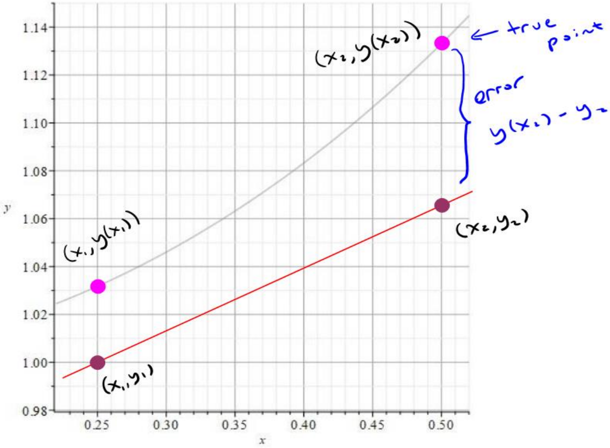

Figure: If we zoom in, we can see that there is some error. But as long as h is small, the point on the tangent line approximates the point on the actual solution curve.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

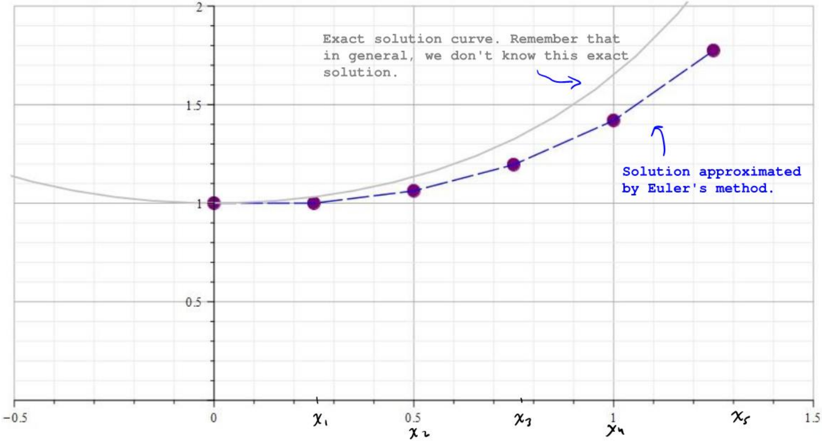

Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution y

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_N, y_N).$$

We'll take the x values to be equally spaced with a common difference of h. That is

$$\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_1 + h = x_0 + 2h \\ x_3 &= x_2 + h = x_0 + 3h \\ &\vdots \\ x_n &= x_0 + nh \end{aligned}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Notation:

- y_n will denote our approximation, and
- $y(x_n)$ will denote the exact solution (that we don't know)

To build a formula for the approximation y_1, let's approximate the derivative at (x_0, y_0).

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope.)

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Let's get a formula for y_1.   Note $x_1 - x_0 = h$

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = f(x_0, y_0)$$

$$\implies \quad y_1 - y_0 = hf(x_0, y_0)$$

So $\boxed{y_1 = y_0 + hf(x_0, y_0)}$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \Longrightarrow \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

Euler's Method Formula: The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

$x_0 = 0$ and $h = 0.25$, so $x_4 = 1$

$x_0 = 0, \quad y_0 = 1 \quad h = 0.25$

$f(x,y) = xy$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + 0.25(0 \cdot 1) = 1$

$x_1 = 0.25, \quad y_1 = 1$

$y_2 = y_1 + h\, f(x_1, y_1) = 1 + 0.25\,(0.25 \cdot 1) = 1.0625$

$x_2 = 0.5, \quad y_2 = 1.0625$

$$y_3 = y_2 + h\,f(x_2, y_2)$$

$$= 1.0625 + 0.25\,(0.5 \cdot 1.0625) = 1.19531$$

$x_3 = 0.75, \quad y_3 = 1.19531$

$$y_4 = y_3 + hf\,(x_3, y_3)$$

$$= 1.19531 + 0.25\,(0.75 \cdot 1.19531)$$

$$= 1.41943$$

$$y_4 = 1.41913 \approx y(1).$$

The true $y(1) = \sqrt{e} \approx 1.64872$