August 23 Math 2306 sec. 51 Spring 2023

Section 2: Initial Value Problems

Initial Value Problem

Recall that an initial value problem, a.k.a an IVP, consists of a differential equation

$$\frac{d^ny}{dx^n} = f(x, y, y', \dots, y^{(n-1)}) \tag{1}$$

subject to a set of *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

Remark: Often, we try to characterize all possible solution to the ODE. Then find parameter values that satisfy the IC.

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

In later sections, we'll have methods for solving some first order ODEs by hand. Here, we look at a method for *approximating* the solution called **Euler's Method**. The idea is simple

- Start with the point (x_0, y_0) that is given,
- use the ODE to make a tangent line $L(x)$ at (x_0, y_0),
- increment the independent variable to a new point x_1
- approximate the solution y using the tangent line, $y(x_1) \approx y_1 = L(x_1)$,
- rinse and repeat!

Euler's Method

Euler's Method Formula:

Consider the first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP, $x_n = x_{n-1} + h$, and h the choice of step size.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Taking a step size of $h = 0.25$, we went through this process and found that $y_4 = 1.41943$ was our approximation to $y(1)$.

The actual[1] solution value $y(1) = \sqrt{e} = 1.64872$. This raises the question of how good our approximation can be expected to be.

[1] The exact solution $y = e^{x^2/2}$.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

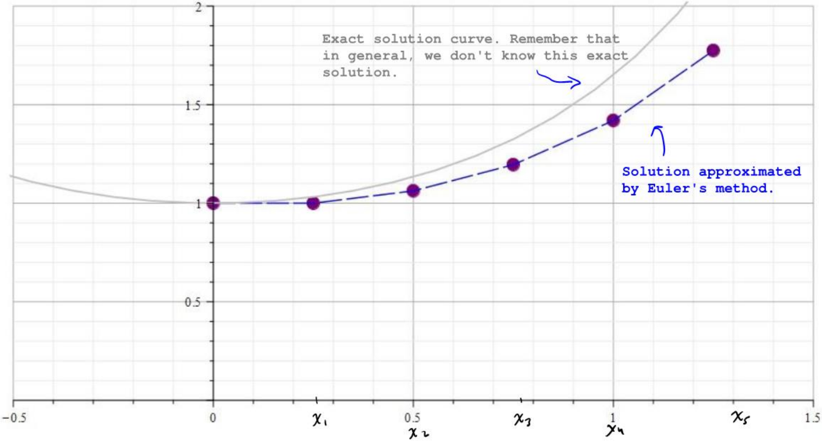

Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: Error

As the previous examples suggest, the approximate solution obtained using Euler's method has error. Moreover, the error can be expected to become more pronounced, the farther away from the initial condition we get.

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[2]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

[2]Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We can ask, how does the error depend on the step size?

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different h values to approximate $y(1)$, and recorded the results shown in the table.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

$\boxed{\text{The absolute error in Euler's method is proportional to the step size.}}$

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- The error in approximating the curve with a tangent line, and
- using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Euler's method is an order 1 scheme.

Other, more accurate methods exist that use some sort of average of *multiple tangent lines*.

- *Improved Euler's Method* is an order 2 method
- *Runge Kutta* is an order 4 method

Euler's Method Example: $\frac{dx}{dt} = \frac{x^2 - t^2}{xt}, \quad x(1) = 2$

Problem: Using a step size of $h = 0.2$, use Euler's method to approximate $x(1.4)$.

$f(t,x) = \frac{x^2 - t^2}{xt}$ $\quad t_0 = 1 \quad x_0 = 2 \quad h = 0.2$

$t_0 = 1, \quad t_1 = 1.2, \quad t_2 = 1.4$ $\quad$ we need x_2.

$$x_1 = x_0 + h f(t_0, x_0)$$

$$= 2 + 0.2\left(\frac{2^2 - 1^2}{2 \cdot 1}\right) \approx 2.3$$

$$x_2 = x_1 + h f(t_1, x_1) \qquad t_1 = 1.2$$

Euler's Method Example: $\frac{dx}{dt} = \frac{x^2 - t^2}{xt}, \quad x(1) = 2$

$$= 2.3 + 0.2\left(\frac{2.3^2 - 1.2^2}{(2.3)(1.2)}\right) \approx 2.57899$$

$$x(1.4) \approx x_2 = 2.57899$$

The true $x(t) = \sqrt{4t^2 - 2t^2 \ln t}$

$$x(1.4) = 2.554$$

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

It is possible to solve this IVP exactly to obtain the solution $x = \sqrt{4t^2 - 2t^2\ln(t)}$. The true value $x(1.4) = 2.554$ to four decimal digits.

Existence and Uniqueness

Existence & Uniqueness Questions

Two important questions we can always pose (and sometimes answer) are

1. Does an IVP have a solution? (existence) and
2. If it does, is there just one? (uniqueness)

As a silly example, consider whether the following can be solved[3]

$$\underbrace{\left(\frac{dy}{dx}\right)^2}_{\geq 1} + 1 = \underbrace{-y^2}_{\leq 0}.$$

[3] If we only wish to consider real valued functions.

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Exercise 1: Verify that $y = \frac{x^4}{16}$ is a solution of the IVP.

Exercise 2: Can you find a second solution of the IVP by inspection—i.e. by clever guessing? (Hint: What's the simplest type of function you can think of. Is there one of that type that satisfies both the ODE and the initial condition?)

This IVP has two distinct solutions. We'll see how to solve the ODE in the next section. The solution technique will give us a 1-parameter family of solutions. We'll find that one of the solutions is a member of the family, and one is not.

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$y = \int \frac{dy}{dx}\,dx$

$= \int (4e^{2x} + 1)\,dx = 2e^{2x} + x + C$

we get a 1-parameter family of solutions

$y = 2e^{2x} + x + C$

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

This is separable w/

$g(x) = x^3, \quad h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

This is not separable

$\dfrac{dy}{dx} = x\left(2 + \dfrac{y}{x}\right)$

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

This is not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$

$$\frac{dy}{dt} = te^{t-y} = te^{t}e^{-y}$$

Is separable w/ $g(t) = te^{t}$

$h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

Recall if $y = f(x)$ $dy = \frac{dy}{dx}\,dx$

$$\Longrightarrow \quad y = G(x) + C$$

where $G'(x) = g(x)$

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables**.

$$\frac{dy}{dx} = g(x)h(y)$$

- Divide by $h(y)$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

- multiply by dx

$$p(y)\underbrace{\frac{dy}{dx}dx}_{dy} = g(x)\,dx$$

we have $p(y)\,dy = g(x)\,dx$

Integrate $\int p(y)\,dy = \int g(x)\,dx$

$$\boxed{\text{we get an implicit solution} \quad P(y) = G(x) + C}$$

where $P'(y) = p(y)$ and $G'(x) = g(x)$