August 23 Math 2306 sec. 53 Fall 2024

Section 3: Separation of Variables

In this and the next section, we'll learn about select types of first order ODEs and how to solve them. The first type is called *separable*. Let's define this and then see how to solve such equations.

Definition:

The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

Remark: Note that the right side is a product with one factor depending only on the independent variables, and one factor depending only on the dependenet variable.

$$\frac{dy}{dx} \stackrel{?}{=} g(x)h(y)$$

Identify each ODE as being separable or not separable.

(a) $\frac{dy}{dx} = x^2y^3$,

separable

$g(x) = x^2$

$h(y) = y^3$.

(b) $\frac{dy}{dx} = 2x + 3y$,

Not separable

(c) $\frac{dy}{dx} = \sin(x^2y)$,

Not separable

(d) $\frac{dy}{dt} = te^{t-y}$

$= te^t \cdot e^{-y}$

separable

$g(t) = te^t$.

$h(y) = e^{-y}$

The Simplest Separable ODE

The simplest type of a separable ODE is one for which the factor $h(y) = 1$. While not very complex, we can generalize from the solution process. Note that

$$\frac{dy}{dx} = g(x)$$

has one parameter family of solutions

$$y = G(x) + C, \quad \text{where} \quad g(x) = G'(x).$$

Example: Solve $\frac{dy}{dx} = e^{4x}$. $\Rightarrow$ $\underbrace{\frac{dy}{dx}\, dx}_{dy} = e^{4x}\, dx$

$\int dy = \int e^{4x}\, dx \Rightarrow y = \frac{1}{4} e^{4x} + C$

Separation of Variable

Consider $\frac{dy}{dx} = g(x)h(y)$, and assume that $h(y) \neq 0$ on the domain of the solution.

We'll obtain an implicit solution by *separating the variables*.

$$\frac{dy}{dx} = g(x)\,h(y)$$ Divide by $h(y)$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$ multiply by dx

$$\frac{1}{h(y)}\underbrace{\frac{dy}{dx}\,dx}_{dy} = g(x)\,dx$$

Recall: If y is a differentiable function of x, then the differential dx is an independent variable and the differential $dy = \frac{dy}{dx}dx$.

$$\frac{1}{h(y)}\,dy = g(x)\,dx \qquad \text{Let } p(y) = \frac{1}{h(y)}$$

$$\int p(y)\,dy = \int g(x)\,dx$$

$$\boxed{P(y) = G(x) + C}$$

where $P'(y) = p(y)$

$G'(x) = g(x)$

This is a one parameter family of solutions defined implicitly.

Find all solutions of the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right)$$ separable w/ $g(x) = -x$, $h(y) = \frac{1}{y}$

Divide by $\frac{1}{y}$

$$y\frac{dy}{dx} = -x$$ mult. by dx

$$y\frac{dy}{dx}\,dx = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

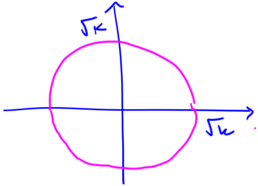
$$\frac{1}{2}y^2 = \frac{-1}{2}x^2 + C$$

This is a 1-parameter family of implicit solns.

Mult. by 2 and set $k = 2C$

$$y^2 = -x^2 + k \implies x^2 + y^2 = k$$

The solutions are circles centered @ $(0,0)$.



Explicit solutions are

$$y = \sqrt{k - x^2}$$

or

$$y = -\sqrt{k - x^2}$$

Example

Let's find an explicit solution to the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0.$$

The ODE is separable w/ $g(x) = x$, $h(y) = \sqrt{y}$.

Divide by $\sqrt{y}$ and mult. by dx

$$\frac{1}{\sqrt{y}} \frac{dy}{dx} dx = x\, dx$$

$$\frac{1}{\sqrt{y}} dy = x\, dx$$

$$\int y^{-1/2} dy = \int x\, dx$$

$$2y^{1/2} = \frac{1}{2}x^2 + C$$

Let's isolate y.

$$y^{1/2} = \frac{1}{4}x^2 + \frac{1}{2}C \qquad \text{let } k = \frac{1}{2}C$$

$$y^{1/2} = \frac{x^2}{4} + k \qquad \text{square}$$

$$y = \left(\frac{x^2}{4} + k\right)^2$$

Apply $y(0) = 0$.

$$y(0) = \left(\frac{0^2}{4} + k\right)^2 = 0$$

$$k^2 = 0 \Rightarrow k = 0.$$

The solution to the IVP is

$$y = \left(\frac{x^2}{4} + 0\right)^2$$

$$y = \frac{x^4}{16}$$

$\dfrac{dy}{dx} = g(x)h(y)$ Caveat regarding division by $h(y)$.

Separation of variables on the ODE $y' = x\sqrt{y}$ leads to the family of solutions $y = \left(\dfrac{x^2}{4} + \dfrac{C}{2}\right)^2$.

The IVP $\boxed{\dfrac{dy}{dx} = x\sqrt{y},\ y(0) = 0}$ has two distinct solutions

$$(1) \quad y = \frac{x^4}{16}, \qquad \text{and} \qquad (2) \quad y = 0.$$

(1) is a member of the family, but (2) is not! That is, the solution (2) can't be found by separation of variables!

Can you identify why we lost the second solution?

we divided by $\sqrt{y}$ assuming it was not zero.

Missed Solutions $\dfrac{dy}{dx} = g(x)h(y)$.

We can state the following theorem about possible missed, constant solutions to separable ODEs.

Theorem:

If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\dfrac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables by looking for solutions to the equation $h(y) = 0$.

Solutions Defined by Integrals

Recall the Fundamental Theorem of Calculus: Suppose g and $\frac{dy}{dx}$ are continuous on some interval $[x_0, b)$ containing x, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x), \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$