August 24 Math 2306 sec. 51 Fall 2022

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x}+1. \qquad \Rightarrow \qquad \underbrace{\frac{dy}{dx}\,dx}_{dy} = (4e^{2x}+1)\,dx$$

$$y = \int (4e^{2x}+1)\,dx = 2e^{2x}+x+C$$

$y = 2e^{2x}+x+C$ one parameter family of solutions

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

Yes $\quad g(x) = x^3$ and

$h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

Not separable

(c) $\dfrac{dy}{dx} = \sin(xy^2)$ Not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ $\dfrac{dy}{dt} = te^{t-y} = te^{t}e^{-y}$

Yes, $g(t) = te^{t}$ and $h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

Recall that if y is a differentiable function of x, then the differential $dy = \dfrac{dy}{dx}dx$. We have

$$\int dy = \int g(x)\,dx \quad \Longrightarrow \quad y = G(x) + C$$

where G is an antiderivative of g.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$

① Divide by $h(y)$

② Multiply by dx use $dy = \frac{dy}{dx}\,dx$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\underbrace{\frac{1}{h(y)}}_{p}\underbrace{\frac{dy}{dx}dx}_{dy} = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx \qquad \text{Integrate}$$

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

This is a one parameter family of solutions defined implicitly.