August 24 Math 2306 sec. 52 Fall 2022

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$\int \frac{dy}{dx}\,dx = \int (4e^{2x} + 1)\,dx$$

$$y = 2e^{2x} + x + C$$

we get a 1-parameter family of solutions

Separable Equations

Definition: The first order equation $y' = f(x,y)$ is said to be **separable** if the right side has the form

$$f(x,y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Determine which (if any) of the following are separable.

(a) $\dfrac{dy}{dx} = x^3 y$

yes with

$g(x) = x^3$, $h(y) = y$

(b) $\dfrac{dy}{dx} = 2x + y$

Not separable

(c) $\dfrac{dy}{dx} = \sin(xy^2)$

Not separable

(d) $\dfrac{dy}{dt} - te^{t-y} = 0$ $\Longrightarrow$ $\dfrac{dy}{dt} = te^{t-y} = te^{t}e^{-y}$

It is separable w/ $g(t) = te^{t}$ and

$h(y) = e^{-y}$

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \frac{dy}{dx}\,dx = \int g(x)\,dx.$$

Recall that if y is a differentiable function of x, then the differential $dy = \dfrac{dy}{dx}dx$. We have

$$\int dy = \int g(x)\,dx \quad \Longrightarrow \quad y = G(x) + C$$

where G is an antiderivative of g.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$

① Divide by $h(y)$

② multiply by dx

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\underbrace{\frac{1}{h(y)}}_{p(y)}\underbrace{\frac{dy}{dx}\,dx}_{dy} = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$

Integrate

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

a 1-parameter family of solutions defined implicitly.

$*$ $P'(y) = p(y)$ and $G'(x) = g(x)$

An IVP[1]

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2$$

$$\frac{1}{Q-70}\frac{dQ}{dt}\,dt = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

$$\ln|Q-70| = -2t + C$$

$$\frac{dQ}{dt} = g(t)\,h(Q)$$

$g(t) = -2,$

$h(Q) = Q-70$

1-parameter family

implicit

[1] Recall IVP stands for *initial value problem*.

Apply: $Q(0) = 180$

$$\ln|180 - 70| = -2(0) + C$$

$$\ln|110| = C \qquad C = \ln(110)$$

The solution is

$$\ln|Q - 70| = -2t + \ln(110)$$ implicit

$$Q(t) = 70 + 110e^{-2t}$$ explicit