August 25 Math 2306 sec. 51 Spring 2023

Section 3: Separation of Variables

Definition:

The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

Remark: Note that the right side is a product with one factor depending only on the independent variables, and one factor depending only on the dependenet variable.

Solving Separable Equations

We found solutions of $\dfrac{dy}{dx} = g(x)h(y)$ by **separating the variables.**

Letting $p(y) = \frac{1}{h(y)}$, we divide[1] by $h(y)$ and multiply by dx.

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x) \quad\Longrightarrow\quad p(y)\underbrace{\frac{dy}{dx}\,dx}_{dy} = g(x)\,dx.$$

Then integrate, $\displaystyle\int p(y)\,dy = \int g(x)\,dx$, to get a 1-parameter family of implicit solutions.

$$\boxed{P(y) = G(x) + c}$$

Here, P and G are any antiderivatives of p and g, respectively.

[1] We'll circle back to this move.

Find all solutions of the ODE

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right)$$

This is separable, so we'll separate variables

$$y\frac{dy}{dx} = -x \quad \Rightarrow \quad \underbrace{y\frac{dy}{dx}\,dx}_{dy} = -x\,dx$$

$$\int y\,dy = \int -x\,dx \quad \Rightarrow \quad \boxed{\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C}$$

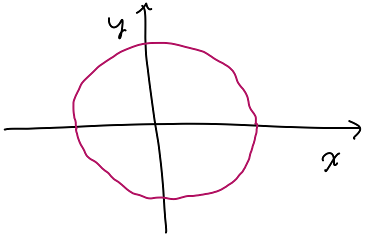
→ This is an 1-parameter family of implicitly defined solutions.

Let's rearrange and find explicit solution(s) multiply by 2 and let $k = 2C$

$$y^2 = -x^2 + k \quad \Rightarrow \quad x^2 + y^2 = k$$

The solutions are circles centered at the origin. We get two explicit solutions

$$y = \sqrt{k - x^2} \quad \text{or} \quad y = -\sqrt{k - x^2}$$



Example

Let's find an explicit solution to the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0.$$

The ODE is separable w/ $h(y) = \sqrt{y}$, $g(x) = x$

Separate variables

$$\frac{1}{\sqrt{y}} \frac{dy}{dx} = x \quad \Rightarrow \quad \frac{1}{\sqrt{y}} \frac{dy}{dx} dx = x\, dx$$

$$\int y^{-1/2}\, dy = \int x\, dx \quad \Rightarrow \quad \frac{y^{1/2}}{1/2} = \frac{1}{2}x^2 + C$$

implicit solution

Let's solve for y.

$$y^{1/2} = \frac{1}{4}x^2 + \frac{1}{2}C \qquad \text{Let} \quad k = \frac{1}{2}C$$

$$\sqrt{y} = \frac{x^2}{4} + k \quad \Longrightarrow \quad y = \left(\frac{x^2}{4} + k\right)^2$$

This is the 1-parameter family of explicit solutions

Apply $y(\overset{x}{0}) = \overset{y}{0}$

$$y(0) = \left(\frac{0^2}{4} + k\right)^2 = 0 \quad \Rightarrow \quad k^2 = 0 \quad \Rightarrow \quad k = 0$$

So $y = \left(\frac{x^2}{4} + 0\right)^2 = \left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}$

The solution to the IVP is

$$y = \frac{x^4}{16}$$

$\frac{dy}{dx} = g(x)h(y)$ Caveat regarding division by $h(y)$.

Separation of variables on the ODE $y' = x\sqrt{y}$ leads to the family of solutions $y = \left(\frac{x^2}{4} + \frac{C}{2}\right)^2$.

The IVP $\boxed{\frac{dy}{dx} = x\sqrt{y},\ y(0) = 0}$ has two distinct solutions

$$(1)\quad y = \frac{x^{\cancel{2}\,4}}{16}, \qquad \text{and} \qquad (2)\quad y = 0.$$

(1) is a member of the family, but (2) is not! That is, the solution (2) can't be found by separation of variables!

Can you identify why we lost the second solution?

we divided by $\sqrt{y}$

This assumes $y \neq 0$!

Missed Solutions $\frac{dy}{dx} = g(x)h(y)$.

We can state the following theorem about possible missed, constant solutions to separable ODEs.

Theorem:

If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\frac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables by looking for solutions to the equation $h(y) = 0$.

Solutions Defined by Integrals

Recall the Fundamental Theorem of Calculus: Suppose g and $\frac{dy}{dx}$ are continuous on some interval $[x_0, b)$ containing x, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x), \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Solutions Defined by Integrals

Show that

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt \quad \text{solves} \quad \frac{dy}{dx} = g(x), \quad y(x_0) = y_0.$$

Let's show that y satisfies the ODE and the I.C.

I.C. $\quad y(x_0) = y_0 + \int_{x_0}^{x_0} g(t)\,dt = y_0 + 0 = y_0$

$y(x_0) = y_0$ ✓

ODE $\quad \frac{d}{dx} y = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} g(t)\,dt\right)$

$$= 0 + \frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \qquad \text{FTC}$$

$$\frac{dy}{dx} = g(x)$$

y solves the IVP.

Generalizing

If p and g are sufficiently continuous then

$$\int_{y_0}^{y} p(z)\,dz = \int_{x_0}^{x} g(t)\,dt \quad \text{solves} \quad \frac{dy}{dx} = \frac{g(x)}{p(y)}, \quad y(x_0) = y_0$$