August 23 Math 2306 sec. 52 Fall 2021

Section 2: Initial Value Problems

We'll recall that **Euler's Method** is a way of approximating the solution to a first order IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Euler's Method Formula: The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

The value $y_n \approx y(x_n)$ where $y(x_n)$ is the true solution to the IVP at $x = x_n$.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Take $h = 0.25$ to find an approximation to $y(1)$.

We went through this process and found that $y_4 = 1.41943$ was our approximation to $y(1)$.

The true[1] $y(1) = \sqrt{e} = 1.64872$. This raises the question of how good our approximation can be expected to be.

[1] The exact solution $y = e^{x^2/2}$.

Euler's Method: Error

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[2]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

[2]Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We can ask, how does the error depend on the step size?

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different h values to approximate $y(1)$, and recorded the results shown in the table.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

$\boxed{\text{The absolute error in Euler's method is proportional to the step size.}}$

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- The error in approximating the curve with a tangent line, and
- using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Euler's method is an order 1 scheme.

Section 3: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

For example, solve the ODE

$$\frac{dy}{dx} = 4e^{2x} + 1.$$

$$y = \int (4e^{2x} + 1)\, dx$$

$$= 2e^{2x} + x + C$$

$$\boxed{y = 2e^{2x} + x + C}$$

a 1-parameter family of solutions.

Separable Equations

Definition: The first order equation $y' = f(x,y)$ is said to be **separable** if the right side has the form

$$f(x,y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

$\frac{dy}{dx} = g(x)$ is separable with $h(y) = 1$.

Separable Equations

Determine which (if any) of the following are separable.

(a) $\frac{dy}{dx} = x^3y$

It's separable, it looks like $\frac{dy}{dx} = g(x)h(y)$ with $g(x) = x^3$ and $h(y) = y$.

(b) $\frac{dy}{dx} = 2x+y$

Not separable, it can't be written as $\frac{dy}{dx} = g(x)h(y)$.

Solving Separable Equations

Recall that from $\frac{dy}{dx} = g(x)$, we can integrate both sides

$$\int \underbrace{\frac{dy}{dx}\,dx}_{dy} = \int g(x)\,dx.$$

$$\int dy = \int g(x)\,dx$$

$$y = G(x) + C$$

where $G'(x) = g(x)$.

We'll use this observation!

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables.**

$$\frac{dy}{dx} = g(x)h(y)$$

① Divide by $h(y)$

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

calling $\frac{1}{h}$ p, $\quad p(y)\frac{dy}{dx} = g(x)$

② Multiply by dx

$$p(y)\underbrace{\frac{dy}{dx}\,dx}_{dy} = g(x)\,dx$$

$$p(y)\,dy = g(x)\,dx$$

③ Integrate each side

$$\int p(y)\,dy = \int g(x)\,dx$$

$$P(y) = G(x) + C$$

where P and G are any antiderivatives of p and g, respectively. Usually this is implicit.

Solve the ODE

It's separable w/ $g(x) = -x$
$h(y) = \frac{1}{y}$.

$$\frac{dy}{dx} = -\frac{x}{y} = -x\left(\frac{1}{y}\right)$$

$$\frac{1}{\frac{1}{y}}\frac{dy}{dx} = -x \implies y\frac{dy}{dx}\,dx = -x\,dx$$

$$y\,dy = -x\,dx \implies \int y\,dy = \int -x\,dx$$

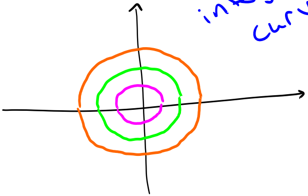
$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} + C}$$

1-parameter family of solutions given implicitly.

$$x^2+y^2=2C$$

$$\boxed{x^2+y^2=k}$$

integral curves



An IVP[3]

$$\frac{dQ}{dt} = g(t)\,h(Q)$$

sure w/ $g(t) = -2$
$h(Q) = Q - 70$

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

We solve the ODE first, then apply the I.C.

$$\frac{1}{Q-70}\frac{dQ}{dt} = -2 \quad \Rightarrow \quad \frac{1}{Q-70}\underbrace{\frac{dQ}{dt}\,dt}_{dQ} = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt$$

$$\ln|Q-70| = -2t + C$$

a 1-parameter family of implicit solns.

Let's get an explicit solution.

[3] Recall IVP stands for *initial value problem*.

I'll solve for Q, then use $Q(0)=180$.

$$e^{\ln|Q-70|} = e^{-2t+C} = e^{-2t}\cdot e^{C}$$

$$|Q-70| = e^{C} e^{-2t} \qquad \text{Let } k = \pm e^{C}$$

$$Q-70 = k e^{-2t} \implies Q = k e^{-2t} + 70$$

Apply $Q(0) = 180$ $\qquad 180 = k e^{0} + 70$

$$k = 180 - 70 = 110$$

$$\boxed{\begin{array}{l}\text{The solution to the IVP is} \\ \qquad Q(t) = 110 e^{-2t} + 70\end{array}}$$

Caveat regarding division by $h(y)$.

The IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

If we separate the variables

$$\frac{1}{\sqrt{y}}\, dy = x\, dx$$

we lose the second solution.

Why? That doesn't make any sense if y is allowed to be zero!