August 25 Math 3260 sec. 53 Fall 2025

1.1.6 Magnitude, Dot Product, and Orthogonality

We saw that nonzero vectors $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$ in R^2 are **parallel** if there exists a scalar c such that $\vec{y} = c\vec{x}$.

The Dot Product

Given the pair of vectors $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$ in R^2, the **dot product** of $\vec{x}$ and $\vec{y}$, denoted $\vec{x} \cdot \vec{y}$, is given by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2.$$

We saw that two nonzero vectors $\vec{x} = \langle x_1, x_2 \rangle$ and $\vec{y} = \langle y_1, y_2 \rangle$ in R^2 are **perpendicular** if there dot product is zero.

Orthogonality

We should observe that for any vector $\vec{x} = \langle x_1, x_2 \rangle$ in R^2, the dot product

$$\vec{x} \cdot \vec{0}_2 = 0.$$

But the zero vector doesn't define an angle with $\vec{x}$. We have a generalization of the notion of perpendicularity.

Orthogonality

We say that two vectors $\vec{x}$ and $\vec{y}$ in R^2 are **orthogonal** if

$$\vec{x} \cdot \vec{y} = 0.$$

Nonzero vectors are perpendicular if they are orthogonal.

Example

Let $\vec{x} = \langle 4, -1 \rangle$. Characterize all vectors in R^2 that are orthogonal to $\vec{x}$.

If $\vec{y} = \langle y_1, y_2 \rangle$ is orthogonal to $\vec{x}$, then

$\vec{x} \cdot \vec{y} = 0$. $\vec{x} \cdot \vec{y}$ $4y_1 + (-1)y_2$. $\vec{y}$ is orthogonal

to $\vec{x}$ if $4y_1 - y_2 = 0$. i.e., $y_1 = \frac{1}{4} y_2$.

Such vectors $y = \langle \frac{1}{4} y_2, y_2 \rangle$.

$$\Rightarrow \vec{y} = y_2 \langle \tfrac{1}{4}, 1 \rangle.$$

All vectors in R^2 that are orthogonal to $\vec{x} = (4, -1)$ are all linear combination of $(\frac{1}{4}, 1)$.

Properties of the Dot Product

The dot product is sometimes called a **scalar product** because it acts on two vectors to produce a scalar. It is an example of something called an **inner product** because it satisfies the following algebraic properties:

For every $\vec{x}$, $\vec{y}$ and $\vec{z}$ in R^2 and scalar c in R

- $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$ (commutative property)
- $(c\vec{x}) \cdot \vec{y} = \vec{x} \cdot (c\vec{y}) = c(\vec{x} \cdot \vec{y})$ (scalars factor)
- $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$ (distributive property)
- $\vec{x} \cdot \vec{x} \geq 0$ with $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}_2$.

It is easy to show that $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$. (The proofs of these statements are homework.)

Section 1.1.7 Direction

Direction Vector

Let $\vec{x} = \langle x_1, x_2 \rangle$ be a nonzero vector. The **direction vector** of $\vec{x}$ is the unit vector

$$\vec{x}_U = \frac{1}{\|\vec{x}\|}\vec{x}.$$

Show that $\vec{x}_U$ is a unit vector.

Recall for vector $\vec{x}$ and scaler c, $\|c\vec{x}\| = |c|\,\|\vec{x}\|$

$$\|\vec{x}_u\| = \left\|\frac{1}{\|\vec{x}\|}\vec{x}\right\| = \left|\frac{1}{\|\vec{x}\|}\right| \|\vec{x}\|$$, but $\|\vec{x}\| > 0$.

$$= \frac{1}{\|\vec{x}\|}\|\vec{x}\| = 1$$

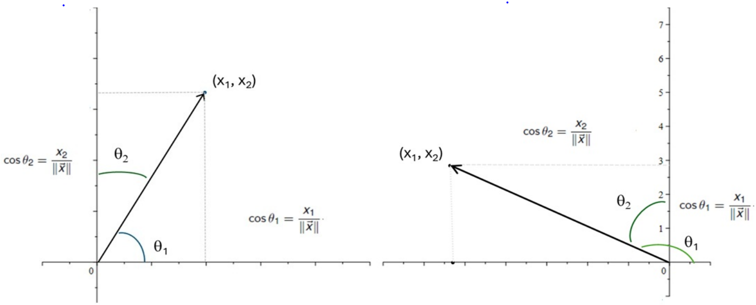

Figure: A nonzero vector makes angles θ_1 and θ_2 relative to the horizontal and vertical, respectively. Note, $0 \le \theta_i \le 180°$.

The numbers

$$\frac{x_1}{\|\vec{x}\|} \quad \text{and} \quad \frac{x_2}{\|\vec{x}\|}$$

are called the **direction cosines** of the vector $\vec{x}$, and the angles

$$\theta_1 = \cos^{-1}\left(\frac{x_1}{\|\vec{x}\|}\right) \quad \text{and} \quad \theta_2 = \cos^{-1}\left(\frac{x_2}{\|\vec{x}\|}\right)$$

are the **direction angles** of the vector $\vec{x}$.

Magnitude × Direction

Let $\vec{x} = \langle x_1, x_2 \rangle \neq \vec{0}_2$. The direction cosines of $\vec{x}$ are

$$\cos\theta_1 = \frac{x_1}{\|\vec{x}\|} \quad \text{and} \quad \cos\theta_2 = \frac{x_2}{\|\vec{x}\|}.$$

Hence the direction vector of $\vec{x}$ is

$$\vec{x}_U = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\|\vec{x}\|}\langle x_1, x_2 \rangle = \left\langle \frac{x_1}{\|\vec{x}\|}, \frac{x_2}{\|\vec{x}\|} \right\rangle = \langle \cos\theta_1, \cos\theta_2 \rangle.$$

Then note that

$$\vec{x} = \langle x_1, x_2 \rangle = \|\vec{x}\| \left\langle \frac{x_1}{\|\vec{x}\|}, \frac{x_2}{\|\vec{x}\|} \right\rangle = \|\vec{x}\| \langle \cos\theta_1, \cos\theta_2 \rangle = \|\vec{x}\|\vec{x}_U.$$

That is,

$$\vec{x} = (\text{magnitude of } \vec{x}) \text{ times } (\text{direction vector of } \vec{x})$$

Example

Find the direction cosines and direction angles of $\vec{x} = \langle -5, 7 \rangle$.

$x_1 = -5$, $x_2 = 7$, $\|\vec{x}\| = \sqrt{(-5)^2 + 7^2} = \sqrt{74}$

The direction cosines are

$\cos\theta_1 = \frac{-5}{\sqrt{74}}$ and $\cos\theta_2 = \frac{7}{\sqrt{74}}$

The direction angles are

$\theta_1 = \cos^{-1}\left(\frac{-5}{\sqrt{74}}\right) \approx 125.5^\circ$

$\theta_2 = \cos^{-1}\left(\frac{7}{\sqrt{74}}\right) \approx 35.5^\circ$

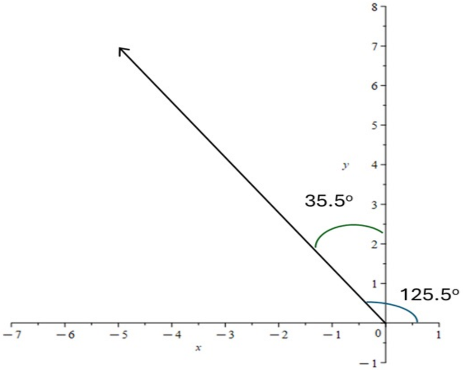

Figure: $\vec{x} = \langle -5, 7 \rangle = \sqrt{74}\langle \cos(125.5^\circ), \cos(35.5^\circ) \rangle$

Section 1.1.8 Distance Between Vectors

Distance Between Vectors in R^2

If $\vec{x}$ and $\vec{y}$ are vectors in R^2, we will denote the distance between the vectors $\mathrm{dist}(\vec{y}, \vec{x})$. This distance,

$$\mathrm{dist}(\vec{y}, \vec{x}) = \|\vec{y} - \vec{x}\|.$$

Note that this gives us the familiar distance formula between the pair of points (x_1, x_2) and (y_1, y_2),

$$\text{Distance} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

seen in an elementary algebra setting.

Distance

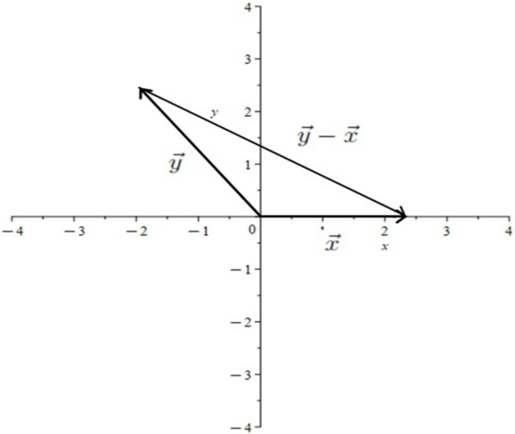

Figure: The distance between two vectors $\vec{x}$ and $\vec{y}$ is the magnitude of their difference. This is the same as the distance between the terminal points of their standard representations.