August 26 Math 2306 sec. 51 Fall 2024

Section 3: Separation of Variables

Separable Differential Equations

Recall that a first order equation of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{1}$$

is called **separable**.

- If $h(c) = 0$, the $y = c$ is a constant solution to (1).
- The equation (1) may be solved by separation of variables,

$$\int \frac{dy}{h(y)} = \int g(x)\, dx$$

Example

Solve the initial value problem $t^2 \dfrac{dx}{dt} = \sec(x), \quad x(1) = 0.$

$\dfrac{dx}{dt} = g(t)\,h(x)$ $\qquad g(t) = \dfrac{1}{t^2}, \quad h(x) = \sec(x)$

$\boxed{\begin{array}{l} \sin(x) = -t^{-1} + 1 \\ \text{is the soln to} \\ \text{the IVP} \end{array}}$

one soln to the ODE is

$x(t) = 1$

$\dfrac{1}{\sec(x)} \dfrac{dx}{dt} = \dfrac{1}{t^2}$

$$t^2\frac{dx}{dt} = \sec(x), \quad x(1) = 0$$

$$\int \cos x \, dx = \int t^{-2} \, dt$$

$$\sin x = -t^{-1} + C$$

Apply $x(1)=0$ i.e. $x=0$ when $t=1$

$$\sin(0) = -1^{-1} + C$$

$$0 = -1 + C \Rightarrow C = 1$$

The solution to the IVP is given implicitly by $\sin x = 1 - \frac{1}{t}$

Solutions Expressed as Integrals

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem $\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$.

Generalizing

If p and g are sufficiently continuous then

$$\int_{y_0}^{y} p(z)\,dz = \int_{x_0}^{x} g(t)\,dt \quad \text{solves} \quad \frac{dy}{dx} = \frac{g(x)}{p(y)}, \quad y(x_0) = y_0$$

Example: Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

Here, $g(x) = \sin(x^2)$, so $g(t) = \sin(t^2)$

$x_0 = \sqrt{\pi}$, $y_0 = 1$

The solution

$$\boxed{y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$$

Section 4: First Order Equations: Linear

Recall that a first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Standard Form

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

$P(x) = \frac{a_0(x)}{a_1(x)}$

$f(x) = \frac{g(x)}{a_1(x)}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

The Solutions of $\dfrac{dy}{dx} + P(x)y = f(x)$

The solution to a first order linear ODE always has the same basic structure

$$y(x) = y_c(x) + y_p(x) \quad \text{where}$$

- $\boxed{y_c}$ is called the **complementary** solution. The complementary solution solves **associated homogeneous** equation, $\dfrac{dy}{dx} + P(x)y = 0$, and
- $\boxed{y_p}$ is called the **particular** solution. The particular solution depends heavily on f and is zero if $f(x) = 0$.

With higher order equations, we'll have to find y_c and y_p separately, but for first order equations we have a process for finding the whole solution.

Motivating Example

Find the solutions of $$x^2\frac{dy}{dx} + 2xy = e^x.$$

This is not in standard form, but we'll work with it as is.

The left side is the derivative of one product; it is $\frac{d}{dx}(x^2 y)$.

Note: $\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy$

so the ODE is

$$\frac{d}{dx}(x^2y) = e^x$$

The goal is to find y. Integrate w/r/t x.

$$\int \frac{d}{dx}(x^2 y)\,dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

$$\Rightarrow \boxed{y = \frac{e^x + C}{x^2}}$$ is a 1-parameter family of solutions

Note: $y = \frac{e^x}{x^2} + \frac{C}{x^2}$

$$y_c(x) = \frac{C}{x^2} \quad \text{and} \quad y_p(x) = \frac{e^x}{x^2}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We'll find a function $\mu(x)$ such that when we multiply through by μ, the left side becomes a product rule.

Assume μ exists and $\mu(x) > 0$ for all x. Multiply by μ

$$\mu(x)\left(\frac{dy}{dx} + P(x)y\right) = \mu(x)\left(f(x)\right)$$

$$\mu \frac{dy}{dx} + P\mu y = \mu f \qquad (1)$$

We want the left side to be $\frac{d}{dx}(\mu y)$.

Note: $$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y \qquad (2)$$

Set the left side of (1) equal to (2)

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + P\mu y$$

This requires $$\frac{d\mu}{dx} y = P\mu y$$

For $y \neq 0$, cancel y to get

$$\frac{d\mu}{dx} = P\mu$$

This is a separable equation for μ.

$$\frac{d\mu}{dx} = P(x)\mu$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\ln \mu = \int P(x)\, dx$$

$$\boxed{\text{The function } \mu = e^{\int P(x)\, dx}}$$

Coming back

$$\mu \frac{dy}{dx} + P\mu y = \mu f$$

$$\frac{d}{dx}(\mu y) = \mu f(x)$$

$$\int \frac{d}{dx}(\mu y)\, dx = \int \mu(x) f(x)\, dx$$

$$\mu y = \int \mu(x) f(x)\, dx + C$$

$$\boxed{y = \frac{1}{\mu} \int \mu(x) f(x)\, dx + \frac{C}{\mu}}$$

Integrating Factor

Integrating Factor

For the first order, linear ODE in standard form

$$\frac{dy}{dx} + P(x)y = f(x),$$

the integrating factor

$$\mu(x) = \exp\left(\int P(x)\,dx\right).$$

Let's list the steps involved in solving a first order linear ODE.

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$