August 26 Math 2306 sec. 52 Fall 2022

Section 3: Separation of Variables

We said that a first order equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called a **separable** differential equation.

A solution (usually implicit) is found by **separating the variables**.

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

(assuming $h(y)$ is nonzero on the interval of interest)

Solve the IVP.

$$\frac{dr}{d\theta} = \frac{\cos\theta}{2r-1}, \quad r\left(\frac{\pi}{2}\right) = 1$$

$$\frac{dr}{d\theta} = g(\theta)\,h(r)$$

$$g(\theta) = \cos\theta$$

$$h(r) = \frac{1}{2r-1}$$

$$(2r-1)\frac{dr}{d\theta} = \cos\theta$$

$$(2r-1)\frac{dr}{d\theta}\,d\theta = \cos\theta\,d\theta$$

$$\int (2r-1)\,dr = \int \cos\theta\,d\theta$$

$$r^2 - r = \sin\theta + C$$

1 parameter family

implicit

Apply $r\left(\frac{\pi}{2}\right) = 1$

$$1^2 - 1 = \sin\left(\frac{\pi}{2}\right) + C$$

$$0 = 1 + C \Rightarrow C = -1$$

The solution to the IVP is given implicitly by

$$r^2 - r = \sin\theta - 1$$

Find an explicit solution to the IVP.

$$\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$$

$g(x) = 4x \quad h(y) = \sqrt{y}$

$\frac{1}{\sqrt{y}} \frac{dy}{dx} = 4x$

$\frac{1}{\sqrt{y}} \frac{dy}{dx} dx = 4x\,dx$

$\frac{1}{\sqrt{y}} dy = 4x\,dx \Longrightarrow \int y^{-1/2} dy = \int 4x\,dx$

$\frac{y^{1/2}}{1/2} = 2x^2 + C \Longrightarrow \sqrt{y} = x^2 + \frac{1}{2}C$

Let $k := \frac{1}{2}C$

A 1-parameter family of solutions is

$$\sqrt{y} = x^2 + k$$

Apply $y(0) = 0$ $\quad \sqrt{0} = 0^2 + k \Rightarrow k = 0$

The solution to the IVP is

$$\sqrt{y} = x^2 \quad \text{(implicit)}$$

To get an explicit solution, square

$$(\sqrt{y})^2 = (x^2)^2 \Rightarrow y = x^4$$

The solution to the IVP is $y = x^4$

Missed Solution

We made an assumption about being able to divide by $h(y)$ when solving $\frac{dy}{dx} = g(x)h(y)$. This may cause us to not find valid solutions.

The IVP $\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$ has two distinct solutions

$$y = x^4, \quad \text{and} \quad y(x) = 0.$$

The second solution **CANNOT** be found by separation of variables. **Why?**

$\frac{1}{\sqrt{y}}$ isn't defined if $y=0$

Missed Solutions $\dfrac{dy}{dx} = g(x)h(y)$.

Theorem: If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\dfrac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables.

Solutions Defined by Integrals

The Fundamental Theorem of Calculus tells us that: If g and $\frac{dy}{dx}$ are continuous on an interval $[x_0, b)$ and x is in this interval, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$g(x) = \sin(x^2)$

$x_0 = \sqrt{\pi}$, $y_0 = 1$

The solution

$$\boxed{y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt}$$

Let's verify that this solves the IVP.

Does $y(\sqrt{\pi}) = 1$?

$$y(\sqrt{\pi}) = 1 + \int_{\sqrt{\pi}}^{\sqrt{\pi}} \sin(t^2)\,dt = 1 + 0 = 1$$

It does satisfy the IC.

Does $\frac{dy}{dx} = \sin(x^2)$?

$$\frac{dy}{dx} = \frac{d}{dx}\left(1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}\int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt$$

$$= 0 + \sin(x^2)$$

$$= \sin(x^2)$$

Yes, it satisfies the ODE