August 27 Math 2306 sec. 51 Fall 2021

Section 3: Separation of Variables

Recall that a first order ODE is called **separable** if it has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Such an equation is solved by **separating the variables.**

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

We can also look for possible constant solutions to the separable ODE by trying to solve the equation

$$h(y) = 0.$$

Caveat regarding division by $h(y)$.

The IVP $\quad \dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$

has two solutions

$$y(x) = \frac{x^4}{16} \quad \text{and} \quad y(x) = 0.$$

The solution $y = x^4/16$ can be obtained by separation of variables.
The constant solution $y(x) = 0$ **cannot**!

Example

In Class Exercise: Take a few minutes and solve the ODE

$$\frac{dy}{dx} = x\sqrt{y}$$

by separating the variables.

If you finish early, try imposing the condition $y(0) = 0$.

$y = \frac{x^4}{16}$

soln to IVP

$$\int y^{-1/2}\,dy = \int x\,dx$$

$$2\sqrt{y} = \frac{x^2}{2} + C$$

implicit family

$$y = \left(\frac{x^2}{4} + k\right)^2 \qquad k = \frac{c}{2}$$

↑ explicit family.

$y(x) = 0$ also solves $\frac{dy}{dx} = x\sqrt{y}$, $y(0) = 0$.

Solutions Defined by Integrals

The Fundamental Theorem of Calculus tells us that: If g and $\frac{dy}{dx}$ are continuous on an interval $[x_0, b)$ and x is in this interval, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$$y = y_0 + \int_{x_0}^{x} g(t)\, dt$$

Let's identify the pieces.

$$g(x) = \sin(x^2), \quad x_0 = \sqrt{\pi}, \quad y_0 = 1$$

$$\boxed{\text{The solution } \quad y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt}$$

Let's verify that it solves

$$\frac{dy}{dx} = \sin(x^2) \quad , \quad y(\sqrt{\pi}) = 1.$$

Show $y(\sqrt{\pi}) = 1$: $\quad y(\sqrt{\pi}) = 1 + \int_{\sqrt{\pi}}^{\sqrt{\pi}} \sin(t^2)\,dt$

$$= 1 + 0 = 1$$

Show $\frac{dy}{dx} = \sin(x^2)$

$$\frac{d}{dx} y = \frac{d}{dx}\left(1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx} \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt$$

$$= 0 + \sin(x^2)$$

$$= \sin(x^2)$$

y satisfies the IC and ODE.

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$P = \frac{a_0}{a_1}$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$f = \frac{g}{a_1}$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy=e^x$$

This is not in standard form, but we'll live with that for this example.

The left hand side (that sum) is the derivative of the product x^2y.

Note $\frac{d}{dx}\left[x^2y\right] = x^2\frac{dy}{dx} + 2xy$

So the ODE is

$$\frac{d}{dx}\left[x^2y\right] = e^x$$

Now we integrate. (The goal is to find y.)

$$\int \frac{d}{dx}\left[x^2 y\right] dx = \int e^x \, dx$$

$$x^2 y = e^x + C$$

So the solution

$$\boxed{y = \frac{e^x + C}{x^2}}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We'll multiply by a function, $\mu(x)$, so the left hand side becomes the derivative of a product, μy. I'll assume that on the domain of definition, $\mu(x) > 0$.

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu f(x) \quad (*)$$

we want the left side to be

$$\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y \quad (**)$$

Compare (*) and (**).

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu \frac{dy}{dx} + \mu P(x) y$$

This requires

$$\frac{d\mu}{dx} y = \mu P(x) y$$

We get a separable equation for μ

$$\frac{d\mu}{dx} = \mu P(x)$$

Solving for μ

$$\int \frac{1}{\mu}\, d\mu = \int p(x)\, dx$$

$$\ln \mu = \int p(x)\, dx$$

So

$$\mu = e^{\int p(x)\, dx}$$

e to an antiderivative of p

μ is called an integrating factor.

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$